IRAQI Academic Scientific Journals

Samarra Journal of Pure and Applied Science

SJPAS

www.sjpas.com

p ISSN: 2663-7405
e ISSN: 2789-6838

Almost Approximaitly NearlyPrime Submodules

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Article Information
Received: 24/10/2022
Accepted: 27/11/2022
Keywords:
Prime submodules, Nearly prime submodules, app-prime submodules. Alappn-prime submodules.

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Abstract

In this paper, we intend to study the concept of almost approximaitly nearlyprime submodules as new generalization of prime submodules, also generalization of approximaitly and nearly prime submodule. We give same basic properties, characterizations, and examples for this concept.

Introduction

Throughout this paper, all rings are commutative rings with identity, and all modules are (left) unitaryR-module. Aproper submodule X of an R-module U is prime submodule, if for any $ru \in X$, for $r \in$ R, and $u \in U$, implies that either $u \in X$ or $rU \subseteq X$. This concept was first introduced by Dauns in 1978 [4]. One of generalization of this concept was studied as nearly prime, where aproper submodule X of an R-module U is nearlyprime submodule, if for any $ru \in X$, for $r \in$ R, and $u \in U$, implies that either $u \in X + J(U\ )$ or $r \in [X + J(U)\colon_{R} U]$ where $J(U)$ is the Jacobsen radical of U defined by the intersection of each maximal submodules of U [9]. In (2019) another generalization of prime submodule introduced was approximaitly prime submodule, where a proper submodule X of an R-module U is called an approximaitly prime submodule of U if for any $ru \in X$, for $r \in$ R, and $u \in U$, implies that either $u \in X + soc(U)$ or $r \in [X + soc(U)\colon_{R} U]$ where $soc(U)$ is defined to be the intersection of each essential submodule of U [1]. In this research we introduce and study a new generalization called almost approximaitly nearlyprime submodules, where aproper submodule X of an R-module U is almost approximaitly nearlyprim submodule, if for any $ru \in X$, for $r \in$ R, $u \in U$, impales that either $u \in X + (soc(U) + J(U))$ or $rU \subseteq\ X + (soc(U) + J(U))$.

Now, we recalled some concepts that we will be using in this paper. A subset S for ring R is multiplicatively closed subset of R if $ab \in S$ for any $a, b \in S$ and $1 \in S$. And if X is submodule of an R-module U and S is a multiplicatively closed subset of R, then $X(S) = \{x \in U\colon \exists t \in S$ such that $tx \in X\}$ is submodule of U and $X \subseteq\ X(S)$ [8]. An R-module U is semi-simple if and only if $soc(U) = U$ [2]. U is injective if for each monomorphism f from R-module A to R-module B and for each R-homomorphism g from A to R-module U there exists an R-

homomorphism h from B to W such that $h \circ f = g$[2]. An element $x \in U$ is called torsion element of U if $ann_R(x) \neq 0$. The set of all torsion elements of U, which is denoted by $\tau(U)$ is a submodule of U. If $\tau(U) = 0$, then R-module U is said to be torsion free [5]. A submodule X of anR-module U is small if $X + Y = U$ then $X = U$ for any proper submodule $Y$of U [6].

Almost Approximaitly NearlyPrime Submodules:

Definition(2.1): A proper submodule X of an Ɍ-module U is called almost approximaitly nearlyprime (simply Alappn-prime) submodule, if for any $ru \in X$, for $r \in$ Ɍ, $u \in U$, impales that either $u \in X + (soc(U) + J(U))$ or $rU \subseteq X + (soc(U) + J(U))$.

And an idael J of a ring Ɍ is called Alappn-prime ideal of Ɍ if J is Alappn-prime Ɍ-submodule of an Ɍ-module Ɍ.

Remarks and examples(2.2)

1. Let $U = Z_{72}$, Ɍ $= Z$, the submodule $X = \langle\bar{4}\rangle$ is Alappn-prime submodule in Z_{72}. Thus for all $r \in Z$, $u \in Z_{72}$, if $ru \in X$ impales that either $ru \in X + \left(soc(Z_{72}) + J(Z_{72})\right) = \langle\bar{4}\rangle + (\langle\overline{12}\rangle + \langle\bar{6}\rangle) = \langle\bar{2}\rangle$ or $r \in \left[X + \left(soc(Z_{72}) + J(Z_{72})\right):_Z Z_{72}\right] = [\langle\bar{2}\rangle:_Z Z_{72}] = 2Z$. That is if $2.\bar{2} = \bar{4} \in X$, for $2 \in Z$, $\bar{2} \in Z_{72}$, impales that $\bar{2} \in X + \left(soc(Z_{72}) + J(Z_{72})\right) = \langle\bar{2}\rangle$ and $2 \in \left[X + \left(soc(Z_{72}) + J(Z_{72})\right):_Z Z_{72}\right] = 2Z$.
2. It is clear that every prime submodule of U is Alappn-prime submodule, The converse is false as in example:

   Let $U = Z_{72}$, Ɍ $= Z$, the submodule $X = \langle\bar{4}\rangle$ is Alappn-prime submodule in Z_{72}[see (1)]. But X isnot prime submodule of Z_{72}, because $2.\bar{2} \in X$, but $\bar{2} \notin X$ and $2 \notin [X: Z_{72}] = 4Z$.
3. It is clear that every approximaitly prime submodule of U is Alappn-prime submodule, The converse is false as in example:

   Let $U = Z_{72}$, Ɍ $= Z$, and the submodule $X = \langle\bar{4}\rangle$ is Alappn-prime submodule of the Z-module Z_{72}[see (1)]. But X is not approximaitly prime submodule of Z_{72}, because $2.\bar{2} \in X$, but $\bar{2} \notin X + soc(Z_{72}) = \langle\bar{4}\rangle + \langle\overline{12}\rangle = \langle\bar{4}\rangle$ and $2 \notin [X + soc(Z_{72}):_Z Z_{72}] = [\langle\bar{4}\rangle:_Z Z_{72}] = 4Z$.
4. It is clear that every nearly prime submodule of U is Alappn-prime submodule, The converse is false as in example:

   Let $U = Z_{72}$, Ɍ $= Z$, and the submodule $X = \langle\bar{6}\rangle$ is Alappn-prime submodule in Z_{12}. Thus for each $r \in Z$, $u \in Z_{12}$, if $ru \in X$, impales that either $u \in X + \left(soc(Z_{12}) + J(Z_{12})\right) = \langle\bar{6}\rangle + (\langle\bar{2}\rangle + \langle\bar{6}\rangle) = \langle\bar{2}\rangle$ or $r \in \left[X + \left(soc(Z_{12}) + J(Z_{12})\right):_Z Z_{12}\right] = [\langle\bar{2}\rangle:_Z Z_{12}] = 2Z$. That is if $3.\bar{2} \in X$, for $3 \in Z$, $\bar{2} \in Z_{12}$, imples that $\bar{2} \in X + \left(soc(Z_{12}) + J(Z_{12})\right) = \langle\bar{2}\rangle$. But X is not nearly prime submodule in Z_{12}, because $3.\bar{2} \in X$, but $\bar{2} \notin X + J(Z_{12}) = \langle\bar{6}\rangle + \langle\bar{6}\rangle = \langle\bar{6}\rangle$ and $3 \notin [X + J(Z_{12}):_Z Z_{12}] = [\langle\bar{6}\rangle:_Z Z_{12}] = 6Z$.
5. The submodule $\langle\bar{6}\rangle$ of the Z-module Z_{72} is not Alappn-prime submodule, because $2.\bar{3} \in \langle\bar{6}\rangle$, for $2 \in Z$, $\bar{3} \in Z_{72}$ but $3 \notin \langle\bar{6}\rangle + \left(soc(Z_{72}) + J(Z_{72})\right) = \langle\bar{6}\rangle + (\langle\overline{12}\rangle + \langle\bar{6}\rangle) = \langle\bar{6}\rangle$ and $2 \notin \left[\langle\bar{6}\rangle + \left(soc(Z_{72}) + J(Z_{72})\right):_Z Z\right] = [\langle\bar{6}\rangle:_Z Z] = 6Z$.
6. If X is Alappn-prime submodule of U, then $[X:_{Ɍ} U]$ isnot Alappn-prime ideal of Ɍ. The following example explain that:

   Let $U = Z_{72}$, Ɍ $= Z$, the submodule $X = \langle\bar{4}\rangle$. X is Alappn-prime submodule in Z_{72} [see(1)]. But $[X:_Z Z_{72}]$=$[\langle\bar{4}\rangle:_Z Z_{72}] = 4Z$ is not Alappn-prime ideal in Z because $2.2 \in 4Z$ for $2{,}2 \in Z$

but $2 \notin 4Z + (soc(Z) + J(Z)) = 4Z + (0) = 4Z$ and $2 \notin [4Z + (soc(Z) + J(Z)):_Z Z] = [4Z:_Z Z] = 4Z$.

7. If X and Y are proper submodules of an Ɍ-module U with $Y \subsetneq X$, and X is Alappn-prime submodule of U, then Y is not Alappn-prime submodule of U. The follawing example explain that:

Let $U = Z_{72}$, Ɍ $= Z$, the submodule $X = \langle\bar{2}\rangle$ is Alappn-prime submodule of Z_{72}(since it is prime) and $Y = \langle\bar{6}\rangle$ is submodule of Z_{72} such that $Y \subsetneq X$, but Y is not Alappn-prime submodule of Z_{72} see(5).

Proposition(2.3). Let U is an Ɍ-module, and X is submodule of U. Then X is Alappn-prime submodule of U if and only if for any submodule Y of U and any idael I of Ɍ with $IY \subseteq X$, impales that either $Y \subseteq X + (soc(U) + J(U))$ or $I \subseteq [X + (soc(U) + J(U)):_{Ɍ} U]$.

Proof: ($\Longmapsto$) Suppose $IY \subseteq X$, for Y is submodule of U and I is an idael of Ɍ, with $Y \nsubseteq X + (soc(U) + J(U))$, then $\exists\ x \in Y$ and $x \notin X + (soc(U) + J(U))$. Since $IY \subseteq X$ then for any $a \in I$, $ax \in X$. But X is Alappn-prime submodule of U and $x \notin X + (soc(U) + J(U))$ then $a \in [X + (soc(U) + J(U)):_{Ɍ} U]$ hence $I \subseteq [X + (soc(U) + J(U)):_{Ɍ} U]$.

($\Longleftarrow$) Let $ru \in X$, for $r \in$ Ɍ, $u \in U$, then $(r)(u) \subseteq X$, so by hypothesis either $(u) \subseteq X + (soc(U) + J(U))$ or $(r) \subseteq [X + (soc(U) + J(U)):_{Ɍ} U]$. That is either $u \in X + (soc(U) + J(U))$ or $r \in [X + (soc(U) + J(U)):_{Ɍ} U]$. Hence X is Alappn-prime submodule of U.

Corollary(2.4). Let U is Ɍ-module ,and X is submodule of U. Then X is Alappn-prime submodule of U if and only if for any submodule Y of U and any $r \in$ Ɍ with $rY \subseteq X$, impales that either $Y \subseteq X + (soc(U) + J(U))$ or $r \in [X + (soc(U) + J(U)):_{Ɍ} U]$.

Proposition(2.5). Let X is a proper submodule of an Ɍ-module U, and $[X + (soc(U) + J(U)):_{Ɍ} U]$ is a prime idael of Ɍ. Then X is Alappn-prime submodule of U if and only if $X(S) \subseteq X + (soc(U) + J(U))$ for each multiplicatively closed subset S of Ɍ such that $S \cap [X + (soc(U) + J(U)):_{Ɍ} U] = \phi$.

Proof: ($\Longmapsto$) Suppose X is Alappn-prime submodule of U , and let $u \in X(S)$, then there exists $s \in S$ such that $su \in X$. But X is Alappn-prime submodule, so either $u \in X + (soc(U) + J(U))$ or $s \in [X + (soc(U) + J(U)):_{Ɍ} U]$. But if $s \in [X + (soc(U) + J(U)):_{Ɍ} U]$, imples that $s \in S \cap [X + (soc(U) + J(U)):_{Ɍ} U] = \phi$, which is a contradiction. Thus $u \in X + (soc(U) + J(U))$ and hence $X(S) \subseteq X + (soc(U) + J(U))$.

($\Longleftarrow$) Let $ru \in X$, for $r \in$ Ɍ, $u \in U$, such that $u \notin X + (soc(U) + J(U))$ and $r \notin [X + (soc(U) + J(U)):_{Ɍ} U]$. Now, since S is a multiplicatively closed subset for Ɍ, then $S = \{1, r, r^2, r^3, \ldots\}$, and since $[X + (soc(U) + J(U)):_{Ɍ} U]$ is a prime idael of Ɍ, then $S \cap [X + (soc(U) + J(U)):_{Ɍ} U] = \phi$. But $u \notin X + (soc(U) + J(U))$, imples that $u \notin X(S)$ and hence $ru \notin X$ which is a contradiction. Thus, either $u \in X + (soc(U) + J(U))$ or $r \in [X + (soc(U) + J(U)):_{Ɍ} U]$, therefore X Alappn-prime submodule of U.

Before we give the next result, we must recall this Lemma.

Lemma(2.6) [3, Theo. (5.1)]: Let I is a proper ideal of ring Ɍ. Then I is maximal idael if and only if $I + \langle a\rangle =$ Ɍ for any $a \notin I$.

Proposition(2.7). Let U is Ɍ-module, and X is submodule of U with $[X + (soc(U) + J(U)):_{Ɍ} U]$ is maximal ideal for Ɍ. Then X is Alappn-prime submodule of U.

Proof: Suppose $ru \in X$, for $r \in \text{R}$, $u \in U$, with $r \notin [X + (soc(U) + J(U)) :_{\text{R}} U]$. Since $[X + (soc(U) + J(U)) :_{\text{R}} U]$ is a maximal ideal of R, then by lemma (2.6) $\text{R} = \langle r \rangle + [X + (soc(U) + J(U)) :_{\text{R}} U]$, where $\langle r \rangle$ is an ideal of R generated by r, we obtain $\exists\, a \in \text{R}$ and $b \in [X + (soc(U) + J(U)) :_{\text{R}} U]$ such that $1 = ar + b$, hence $u = aru + bu \in X + (soc(U) + J(U))$. Hence Alappn-prime submodule of U.

Propsition(2.8). Let U is an R-module, and J is maximal ideal of R, such that $JU + (soc(U) + J(U)) \subsetneq U$. Then JU is Alappn-prime submodule of U.

Proof: Since $JU \subseteq JU + (soc(U) + J(U))$, imples that $J \subseteq [JU + (soc(U) + J(U)) :_{\text{R}} U]$, that is there exists $a \in [JU + (soc(U) + J(U)) :_{\text{R}} U]$ and $a \notin J$, but J is a maximal ideal then by lemma (2.6) $J + \langle a \rangle = \text{R}$, where $\langle a \rangle$ is an ideal of R generated by a, thus there exist $r \in \text{R}$ and $j \in J$ such that $1 = j + ar$, it follows that $u = ju + aru$ for each $u \in U$. Hence $u \in JU + (soc(U) + J(U))$, for each $u \in U$, that is $U \subseteq JU + (soc(U) + J(U))$, hence $U = JU + (soc(U) + J(U))$ which is a contradiction. Then $a \in J$ and hence $[JU + soc(U) :_{\text{R}} U] \subseteq J$, it follows that $[JU + soc(U) :_{\text{R}} U] = J$ is a maximal ideal of R, hence by proposition (2.7) JU is Alappn-prime submodule of U.

Proposition(2.9). Let X is a proper submodule of an R-module U, with $[Y :_{\text{R}} U] \nsubseteq [X + (soc(U) + J(U)) :_{\text{R}} U]$, and $X + (soc(U) + J(U))$ is a proper submodule of Y for each submodule Y of U such that $[X + (soc(U) + J(U)) :_{\text{R}} U]$ is a prime ideal in R. Then X is Alappn-prime submodule of U.

Proof: Suppose $ru \in X$, for $r \in \text{R}$, $u \in U$, and $u \notin X + (soc(U) + J(U))$. Then $X + (soc(U) + J(U)) \subsetneq X + (soc(U) + J(U)) + \langle u \rangle = Y$ and so $[Y :_{\text{R}} U] \nsubseteq [X + (soc(U) + J(U)) :_{\text{R}} U]$, then there exists $a \in [Y :_{\text{R}} U]$ and $a \notin [X + (soc(U) + J(U)) :_{\text{R}} U]$. That is $aU \subseteq Y$ and $aU \nsubseteq X + (soc(U) + J(U))$. Thus $aU \subseteq Y$, imples that $raU \subseteq r(X + (soc(U) + J(U)) + \langle u \rangle) \subseteq X + (soc(U) + J(U))$. It follows that $ra \in [X + (soc(U) + J(U)) :_{\text{R}} U]$. But $[X + (soc(U) + J(U)) :_{\text{R}} U]$ is a prime idael of R, and $a \notin [X + (soc(U) + J(U)) :_{\text{R}} U]$ then $r \in [X + (soc(U) + J(U)) :_{\text{R}} U]$. Hence X is Alappn-prime submodule.

Proposition(2.10). Let U is an R-module, and X, Y are submodules in U with $Y \subseteq X$, and X is a proper submodule of U. If X is Alappn-prime submodule of U, then $\frac{X}{Y}$ is Alappn-prime submodule of $\frac{U}{Y}$.

Proof: Suppose X is Alappn-prime submodule of U, and let $r(u + Y) = ru + Y \in \frac{X}{Y}$, for $r \in \text{R}$, $u + Y \in \frac{U}{Y}$, $u \in U$. Then $ru \in X$. But X is Alappn-prime submodule of U, imples that either $u \in X + (soc(U) + J(U))$ or $rU \subseteq X + (soc(U) + J(U))$. Thus $u + Y \in \frac{X + (soc(U) + J(U))}{Y}$ or $\frac{U}{Y} \subseteq \frac{X + (soc(U) + J(U))}{Y}$, that is either $u + Y \in \frac{X}{Y} + \frac{X + soc(U)}{Y} + \frac{X + J(U)}{Y} \subseteq \frac{X}{Y} + soc\left(\frac{U}{Y}\right) + J\left(\frac{U}{Y}\right)$ or $r\frac{U}{Y} \subseteq \frac{X}{Y} + \frac{X + soc(U)}{Y} + \frac{X + J(U)}{Y} \subseteq \frac{X}{Y} + soc\left(\frac{U}{Y}\right) + J\left(\frac{U}{Y}\right)$. Hence $\frac{X}{Y}$ is Alappn-prime submodule of $\frac{U}{Y}$.

Now before we give the converse of proposition (2.10) we need to recall the following lemma.

Lemma(2.11) [6, Ex.(12). P 239]

a) Let X is a direct summand submodule of an R-module U then $J\left(\frac{U}{X}\right) = \frac{J(U) + X}{X}$

b) An R-module U is semi-simple if and only if for all submdule X of U $soc\left(\frac{U}{X}\right) = \frac{soc(U) + X}{X}$

Proposition(2.12). Let U is semi-simple R-module, and X, Y are submodules in U with $Y \subseteq X$, and X is a proper submodule of U. If Y and $\frac{X}{Y}$ are Alappn-prime submodule of U and $\frac{U}{Y}$ respectively, then X is Alappn-prime submodule of U.

Proof: Suppose Y and $\frac{X}{Y}$ are Alappn-prime submodule of U and $\frac{U}{Y}$ respectively, and let $ru \in X$, for $r \in$ R, $u \in U$. So $(u+Y) = ru + rY \in \frac{X}{Y}$. If $ru \in Y$ and Y is Alappn-prime submodule of U, imples that either $u \in Y + (soc(U) + J(U)) \subseteq X + (soc(U) + J(U))$ or $rU \subseteq Y + (soc(U) + J(U)) \subseteq X + (soc(U) + J(U))$, hence X is Alappn-prime submodule of U. Therefore, we may assume that $ru \notin Y$. Follow it $(u+Y) \in \frac{X}{Y}$ **,** but $\frac{X}{Y}$ is Alappn-prime submodule of $\frac{U}{Y}$**,** implies that either $u + Y \subseteq \frac{X}{Y} + \left(soc\left(\frac{U}{Y}\right) + J\left(\frac{U}{Y}\right)\right)$ or $r\frac{U}{Y} \subseteq \frac{X}{Y} + \left(soc\left(\frac{U}{Y}\right) + J\left(\frac{U}{Y}\right)\right)$. Since U is semi-simple then by lemma (2.11) either $u + Y \in \frac{X}{Y} + \frac{Y+soc(U)}{Y} + \frac{Y+J(U)}{Y}$ or $r\ \frac{U}{Y} \subseteq \frac{X}{Y} + \frac{Y+oc(U)}{Y} + \frac{Y+J(U)}{Y}$. Since $Y \subseteq X$, it follows that $Y + soc(U) \subseteq X + soc(U)$ and $Y + J(U) \subseteq X + J(U)$, hence $\frac{X}{Y} + \frac{Y+soc(U)}{Y} + \frac{Y+J(U)}{Y} \subseteq \frac{X}{Y} + \frac{X+soc(U)}{Y} + \frac{X+J(U)}{Y}$. Thus either $u + Y \in \frac{X+(soc(U)+J(U))}{Y}$ or $r\frac{U}{Y} \subseteq \frac{X+(soc(U)+J(U))}{Y}$, it follows that either $u \in X + (soc(U) + J(U))$ or $r\,U \subseteq X + (soc(U) + J(U))$. Hence X is Alappn-prime submodule of U.

Lemma(2.13) [10, Lem. (2.3)]: Let U be injective R-module then $J(U) = U$.

Proposition(2.14): Every submodule of injective R-module is an Alappn-prime submodule.

Proof: Let X be a submodule of an R-module U, and $ru \in X$, for $r \in$ R, $u \in U$. Since U is injective then by Lemma (2.13) $J(U) = U$. We have $u \in U = J(U) \subseteq X + (soc(U) + J(U))$ or $rU \subseteq U = J(U) \subseteq X + (soc(U) + J(U))$. That is X is an Alappn-prime submodule of U.

Proposition(2.15): If rX is an Alappn-prime submodule of R-module U, for r is idempotent in R and X is submodule of U. Then $X \subseteq rX + (soc(U) + J(U))$ or $X + \left(soc(U) + J(U)\right) = U$.

Proof: Let $u \in X$, then $ru \in rX$, where $r \in$ R. Since rX is an Alappn-prime submodule of U then either $u \in rX + (soc(U) + J(U))$ or $rU \subseteq rX + (soc(U) + J(U))$ that is either $u \in rX + (soc(U) + J(U))$ or $r^2U \subseteq r^2X + r\,soc(U) + r\,J(U)$. But r is an idempotent, so either $u \in rX + (soc(U) + J(U))$ or $rU \subseteq rX + r\,soc(U) + r\,J(U)$. Hence either $u \in rX + (soc(U) + J(U))$ or $U \subseteq X + (soc(U) + J(U))$. But $X \subseteq U$, $soc(U) \subseteq U$, and $J(U) \subseteq U$, then $X + (soc(U) + J(U)) \subseteq U$. Thus $X \subseteq rX + (soc(U) + J(U))$ or $X + \left(soc(U) + J(U)\right) = U$.

Proposition(2.16): Let U be an R-module with every nonzero element of R is invertible. Then every submodule of U is Alappn-prime.

Proof: Let X be a proper submodule of U, and $ru \in X$, for $r \in$ R, $u \in U$. Since r is invertible then there exists $r^{-1} \in$ R such that $r^{-1}ru \in r^{-1}X \subseteq X$ implies that $r^{-1}ru \in X$ that is $1.u \in X \subseteq X + (soc(U) + J(U))$, hence $u \in X \subseteq X + (soc(U) + J(U))$. Therefore X is an Alappn-prime submodule of U.

Corollary(2.17): Let U be an R-module over field. Then every submodule of U is Alappn-prime.

Proposition(2.18): Every submodule of torsion free module is Alappn-prime.

Proof: Let X be a proper submodule of a torsion free R-module U, and $ru \in X$, for $r \in$ R, $u \in U$, then $rru \in rX$ that is $r^2u \in rX$. Thus there exists $n \in X$ such that $r^2u = rn$ it follows that $r[ru - n] = 0$, but $ru \in X \subseteq U$ and $n \in X \subseteq U$ it follows that $ru - n \in U$, thus $r \in ann_{\mathrm{R}}(ru -$

$n)$. Since U is torsion free then $ann_R(ru-n)=0$ it follows that $r=0$, hence $rU=\{0\}\subseteq X+(soc(U)+J(U))$. Thus $rU\subseteq X+(soc(U)+J(U))$. Therefore X is an Alappn-prime submodule of U.

Remark(2.19): The intersection of two Alappn-prime submodules in R-module U isnot Alappn-prime submodule of U. As in this following example:

Let $U=Z_{72}$, $R=Z$. $X=\langle\bar{2}\rangle$ and $Y=\langle\bar{3}\rangle$ are Alappn-prime submodules of Z_{72} (because they are prime), but $X\cap Y=\langle\bar{6}\rangle$ is not Alappn-prime submodule of Z_{72} see [Remarks and Examples (2.2)(5)].

The following proposition shows that the intersection of two Alappn-prime submodules is Alappn-prime submodule under certain condition.

Lemma(2.20) [7, Ex.12(5). p242]. A submodule X of an R-module is maximal and essential if $soc(U)\subseteq X$.

Proposition(2.21). Let U is an R-module with either X or Y are maximal essential submodule in U and $Y\nsubseteq X$. If X and Y are Alappn-prime submodules in U then $X\cap Y$ is Alappn-prime submodule of U.

Proof: Let $ru\in X\cap Y$, for $r\in R$, $u\in U$, then $ru\in X$ and $ru\in Y$. But X and Y are Alappn-prime submodules of U, then either $u\in X+(soc(U)+J(U))$ or $rU\subseteq X+(soc(U)+J(U))$ and $u\in Y+(soc(U)+J(U))$ or $rU\subseteq Y+(soc(U)+J(U))$. Thus $u\in\big(X+(soc(U)+J(U))\big)\cap\big(Y+(soc(U)+J(U))\big)$ or $rU\subseteq\big(X+(soc(U)+J(U))\big)\cap\big(Y+(soc(U)+J(U))\big)$. Since X or Y are maximal essential in U then lemma (2.20) $soc(U)\subseteq X$ or $soc(U)\subseteq Y$ and since X or Y are maximal then $J(U)\subseteq X$ or $J(U)\subseteq Y$. Suppose X is a maximal essential of U then $soc(U)\subseteq X$ and $J(U)\subseteq X$ and hence $(soc(U)+J(U))\subseteq X$, it follows that either $u\in X\cap\big(Y+(soc(U)+J(U))\big)$ or $rU\subseteq X\cap\big(Y+(soc(U)+J(U))\big)$. Therefor by Modular law[6] either $u\in(X\cap Y)+(soc(U)+J(U))$ or $rU\subseteq(X\cap Y)+(X\cap Y)+(soc(U)+J(U))$. Therefore $X\cap Y$ is Alappn-prime submodule of U.

Proposition(2.22). Let $f:U\to U'$ is R-epimorphism and X is Alappn-prime submodule of an R-module U'. Then $f^{-1}(X)$ is Alappn-prime submodule of an R-module U.

Proof: It is clear that $f^{-1}(X)$ is a proper submodule of U. Now, suppose $ru\in f^{-1}(X)$, for $r\in R$, $u\in U$, imples that $rf(u)\in X$. But X is Alappn-prime submodule of U', then either $f(u)\in X+(soc(U')+J(U'))$ or $rU'\subseteq X+(soc(U')+J(U'))$. It follows that either $u\in f^{-1}(X)+f^{-1}\Big(\big(soc(U')+J(U')\big)\Big)\subseteq f^{-1}(X)+(soc(U)+J(U))$ or $rf^{-1}(f(U))\subseteq f^{-1}(X)+f^{-1}\Big(\big(soc(U')+J(U')\big)\Big)\subseteq f^{-1}(X)+(soc(U)+J(U))$. Hence either $u\in f^{-1}(X)+(soc(U)+J(U))$ or $rU\subseteq f^{-1}(X)+(soc(U)+J(U))$. Thus $f^{-1}(X)$ is Alappn-prime submodule of U.

Proposition(2.23). Let $f:U\to U'$ is an R-epimorphism, and $\ker f$ is a small submodule in U. If X is Alappn-prime submodule of U with $\ker f\subseteq X$. Then $f(X)$ is Alappn-prime submodule of U'.

Proof: It is clear that $f(X)$ is a proper submodule of U'. Now let $ru'\in f(X)$, for $r\in R$, $u'\in U'$, imples $rf(u)\in f(X)$ for some $u\in U$ (since f is onto), that is $f(ru)\in f(X)$, $f(ru)=f(x)$ for some $x\in X$ that is $f(x-ru)=0$, so $x-ru\in Ker\,f\subseteq X$, imples that $ru\in X$. But X is Alappn-prime submodule of U, then either $u\in X+(soc(U)+J(U))$ or $rU\subseteq X+(soc(U)+J(U))$. Hence either $u'=f(u)\in f(X)+f\Big(\big(soc(U)+J(U)\big)\Big)\subseteq f(X)+(soc(U')+J(U'))$ or $rU'=$

$rf(U) \subseteq f(X) + f\left(\left(soc(U) + J(U)\right)\right) \subseteq f(X) + (soc(U') + J(U'))$. Thus $f(X)$ is Alappn-prime submodule of U'.

Proposition(2.24). Let $U = U_1 \oplus U_2$ is an R-module and $X = X_1 \oplus X_2$ is a semi-simple Alappn-prime submodule of U, with $X \subseteq J(U)$.Then X_1 and X_2 are Alappn-prime submodules of U_1 and U_2 respectively.

Proof: To prove X_1 is Alappn-prime submodule of U_1. Let $ru_1 \in X_1$, for $r \in$ R, $u_1 \in U_1$, then $r(u_1, 0) \in X_1 \oplus X_2$. Since $X_1 \oplus X_2$ is Alappn-prime submodule of U, then either $(u_1, 0) \in X + (soc(U) + J(U))$ or $r \in [X + (soc(U) + J(U)):_R U]$. But X is semi-simple then $X = soc(X) \subseteq soc(U)$ and since $X \subseteq J(U)$ imples that $X + (soc(U) + J(U)) = soc(U) + J(U)$, hence either $(u_1, 0) \in soc(U) + J(U) = soc(U_1 \oplus U_2) + J(U_1 \oplus U_2) = soc(U_1) \oplus soc(U_2) + J(U_1) \oplus J(U_2) = (soc(U_1) + J(U_1)) \oplus (soc(U_2) + J(U_2))$ or $r \in [soc(U) + J(U):_R U] = \left[\left(soc(U_1) + J(U_1)\right) \oplus \left(soc(U_2) + J(U_2)\right):_R U_1 \oplus U_2\right] \subseteq [soc(U_1) + J(U_1):_R U_1] \subseteq [X_1 + soc(U_1) + J(U_1):_R U_1]$. So either $u_1 \in soc(U_1) + J(U_1) \subseteq X_1 + soc(U_1) + J(U_1)$ or $r \in [X_1 + soc(U_1) + J(U_1):_R U_1]$. Hence X_1 is an Alappn-prime submodule of U_1.

Similarly, we can prove that X_2 is Alappn-prime submodule of U_2.

References

1. Ajeel, A. S., & Ali, H. K. M. (2019). Approximaitly Prime Submodules and Some Related Concepts. *Ibn AL-Haitham Journal for Pure and Applied Science*, *32*(2), 103-113.
2. Anderson, F. W., & Fuller, K. R., (1992). *Rings and Categories of Modules*. Springer- Verlag New York.
3. Burton, D. M., (1970). *A First Course in Rings and Ideals*. University of New Hampshire.
4. Dauns, J. (1978). Prime Modules. *Journal Reine Angew, Math., 2*, 156-181.
5. Goodearl, K. (1976). *Ring theory: Nonsingular rings and modules* (Vol. 33). CRC Press.
6. Kasch, F. (1982). *Modules and rings* (Vol. 17). Academic press.
7. Lam, T.Y. (1999). *Lectures on Modules and Rings*, GTN, 189, Springer Verlage, Isrlin Newyork.
8. Behrens, E. A., & Reis, C. (1971). *Multiplicative theory of ideals*. Academic press.
9. Al-Mothafar, N. S., & Abdula-Al-Kalik, A. J. (2015). Nearly Prime submodules. *international Journal of Advanced Scientific and Technical Research*, *6*(3), 166-173.
10. Majid, M. A. & Ghafur, A. (2012). Generalization of Generalized Supplemented Modules, *International Journal of Algebra*. *6*(29), 1431-1441.

IRAQI Academic Scientific Journals

Samarra Journal of Pure and Applied Science

SJPAS
مجلة سامراء للعلوم الصرفة والتطبيقية

www.sjpas.com

p ISSN: 2663-7405
e ISSN: 2789-6838

المقاسات الجزئية الاولية المتقارب تقريباً كلياً

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البحث مستل من أطروحة الدكتوراه الباحث الاول

معلومات البحث:

تأريخ الاستلام: 2022/10/24
تأريخ القبـــول: 2022/11/27

الكلمات المفتاحية:

المقاسات الجزئيه الاولية، المقاسات الجزئيه المتقاربه. المقاسات الجزئيه الأولية تقريباً. المقاسات الجزئيه الاوليه المتقارب تقريباً كلياً.

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الخلاصة:

تتناول هذه الدراسة مفهوم المقاسات الجزئية الأولية المتقارب تقريباً كلياً كتعميم جديد للمقاسات الجزئية الأولية، وكذلك تعميم للمقاسات الجزئية الأولية تقريباً والأولية المتقاربة. اعطينا بعض الخصائص الاساسية والمتكافئات والأمثلة لهذا المفهوم.