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On Multi -Valued Contractive Mappings and Fixed-Point Theorems in Complete $\mathcal{D}^*$-Symmetric Spaces

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Abstract

The importance of fixed-point theory has encouraged many authors to consider it in various types of single and multi-valued mappings and it is gained tremendous implementations in many fields of pure and applied mathematics and other sciences; also, this theory for multi-valued contraction mappings has acquired formidable implementation in venous sciences such as optimization, economics, deferential equations and control theory. Therefore, the main motivation of present manuscript is to study and verify various fixed-point theorems for multi-valued contraction maps, as well existence of common fixed points and uniqueness for occasionally-weakly compatible maps has been proved under influence some certain generalized multi-valued weak-contractive conditions in the context of new extended symmetric space namely, $\mathcal{D}^*$-symmetric space which is extension of $\mathcal{D}^*$-metric and G-symmetric spaces. Our major outcomes which are related to these kinds of fixed-point theorems for multi-valued contraction maps are extensions of the various outcomes existing in the literature. Additionally, suitable examples that support our major outcomes have been prepared

Introduction

The concept of fixed point theory for single and multi-valued maps satisfying various extended contractive conditions is playing a significant role in various directions to construct various procedures to solve many problems in pure and applied mathematics. J. Dugundji and A. Granas [1] verified single weakly contractive map of metric spaces has unique fixed point. After that, H. Kaneko[2] presented partial extension of the theorem of Dugundji and Granas to multi-valued maps. P. Z. Daffer in [3] studied a number of problems associated with fixed points of weakly contractive multi-valued maps. Subsequently, D. El-Moutawakil [4] gave an original extension of renowned multi-valued contraction fixed point in symmetric-sp. Following, H.Chandra & A.Bhatt [5] verified fixed point theory for extended contraction under restrictive conditions in symmetric-sp. On the other hand, M. Abbas, et al. [6] Established diverse fixed point theories for multivalued maps under extended contractive conditions in ordered generalized metric-sp. Afterward, K. S. Eke and J. O. Olaleru, [7] expand the idea of symmetric-sp to G-Symmetric-sp, as well the existence of common fixed points for (O.W.C) maps is proved in G-Symmetric-sp. Motivated by this fact, in [8] they are verified existence of common fixed to hybrid contractive maps in G-Symmetric-sp. Lately, A. M. Al-Jumaili, et al. [9] extended the conception of $\mathcal{D}^*$-metric-sp by changing R by an ordered Banach-sp in $\mathcal{D}^*$-metric-sp, as well in same year K.S.Eke and J. G.Oghonyon [10] established existence of the common fixed for Hardy & Rogers-kind and Ciric-kind maps in setting of G-symmetric-sp. Newly, N. A. Majid, et al. [11] verified a number of original fixed point outcomes for monotone multi-valued maps in partially ordered complete $\mathcal{D}^*$-metric-sp, as well various existence and uniqueness of coupled fixed point outcomes of maps satisfying contractive condition have been investigated, additionally in [12] they investigated various outcomes of common and coincidence of fixed points in S-metric-sp. Afterward, A. H. Abed & A. M. F. Al-Jumaili, [13] defined novel kind of generalized metric-sp namely, $\mathcal{D}^*$-symmetric-sp and established numerous common-fixed point outcomes for maps satisfying extended contractive conditions of $\mathcal{D}^*$-symmetric-sp. Major purpose of the present work is investigate and verify various fixed point theorems to multi-valued contraction maps in the setting of $\mathcal{D}^*$-symmetric-sp. In addition, our aim of this manuscript is to confirm a number of existences and uniqueness of common fixed point outcomes for (O.W.C) maps under influence multi-valued weak-contractive conditions in context of complete $\mathcal{D}^*$symmetric-sp. Our major results are extensions, generalizations and improvement of the various renowned results in metric space and as well.in the setting of symmetric-sp.

Preliminaries

In this segment, start with various definitions and motivations are needed in the sequel and that will help us in the results that follow and play essential role in this work for verifying our major outcomes.

Definition 2.1: [14] Let $\mathcal{X} = \emptyset$ and $\mathcal{D}^*: \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to [0, \infty)$, be a map satisfying the subsequent conditions $\forall\, u, v, w, z \in \mathcal{X}$:
$(\mathcal{D}^*_1)\mathcal{D}^*(u, v, w) \geq 0, \forall\, u, v, w \in \mathcal{X}$;
$(\mathcal{D}^*_2)\mathcal{D}^*(u, v, w) = 0$ if and only if $u = v = w$;
$(\mathcal{D}^*_3)\mathcal{D}^*(u, v, w) = \mathcal{D}^*(\mathcal{P}\{u, v, w\})$, (symmetry) where $\mathcal{P}$ is a permutation map,
$(\mathcal{D}^*_4)\ \mathcal{D}^*(u, v, w) \leq \mathcal{D}^*(u, v, z) + \mathcal{D}^*(z, w, w)$.
It follows that, $\mathcal{D}^*$ is called $\mathcal{D}^*$-metric and $(\mathcal{X}, \mathcal{D}^*)$ is $\mathcal{D}^*$-metric-sp (briefly. metric-sp).
Example 2.2: [14] The Immediate examples of such a map are follows:
$(\boldsymbol{i})\mathcal{D}^*(u, v, w) = max\{|u - v|, |v - w|, |u - w|\}$,
$(\boldsymbol{ii})\mathcal{D}^*(u, v, w) = |u - v| + |v - w| + |u - w|$.
Definition 2.3: [15] A symmetric on $\mathcal{X}$ is a real valued map d on $\mathcal{X} \times \mathcal{X}$ (s. t):
$(\boldsymbol{d_1})d(u, v) \geq 0, \forall\, u, v \in \mathcal{X}$.
$(\boldsymbol{d_2})d(u, v) = 0 \Leftrightarrow u = v$.
$(\boldsymbol{d_3})d(u, v) = d(v, u)$.
Wilson [16] as well presented some axioms of d-symmetric:
$(\boldsymbol{W_1})$Given $\{u_s\}$ & $u, v \in \mathcal{X}$, $d(u_s, u) \longrightarrow 0$ & $d(u_s, v) \longrightarrow 0 \Longrightarrow u = v$,
$(\boldsymbol{W_2})$Given $\{u_s\}, \{v_s\}$ & $u, v \in \mathcal{X}$, $d(u_s,, u) \longrightarrow 0$ & $(u_s, v_s) \longrightarrow 0 \Longrightarrow \mathcal{D}^*_d(v_s, u) \longrightarrow 0$.
Definition 2.4: [13] A $\mathcal{D}^*$-symmetric on $\mathcal{X}$ is $\mathcal{D}^*_d: \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to [0, \infty)$ (s. t) $\forall u, v, w \in \mathcal{X}$ the next conditions are satisfied:
$(\mathcal{D}^*_{d_1})\mathcal{D}^*{}_d\ (u, v, w) \geq 0, \forall\, u, v, w \in \mathcal{X}$;
$(\mathcal{D}^*_{d_2})\mathcal{D}^*{}_d(u, v, w) = 0 \Leftrightarrow u = v = w$;

$(\mathcal{D}^*_{d_3})\mathcal{D}^*{}_d(u,v,w)=\mathcal{D}^*{}_d(\mathcal{P}\{u,v,w\})$, (symmetry) where $\mathcal{P}$ is a permutation map,

It should be observed that our idea of $\mathcal{D}^*$-symmetric space is the same as that of $\mathcal{D}^*$-metric space **(Definition 2.1)** without the rectangle inequality property-$(\mathcal{D}^*_4)$.

Example 2.5: [13] Assume $\mathcal{X}=[0,1]$ equipped with $\mathcal{D}^*$-symmetric described via $\mathcal{D}^*_d(u,v,w)=(u-v)^2+(v-w)^2+(w-u)^2,\quad \forall\, u,v,w\in\mathcal{X}$. So, $(\mathcal{D}^*_d,\mathcal{X})$ is $\mathcal{D}^*$-symmetric-sp. This doesn't satisfy the rectangle inequality of $\mathcal{D}^*$-metric-sp, for this reason it's not $\mathcal{D}^*$-metric-sp.

The analogue of axioms of Wilson [16] in $\mathcal{D}^*$-symmetric-sp is follows:

(W_3) Given $\{u_s\}, u,v\in\mathcal{X}$, $\mathcal{D}^*_d(u_s,u,u)\longrightarrow 0$ & $\mathcal{D}^*_d(u_s,v,v)\longrightarrow 0\Longrightarrow u=v$.

(W_4) Given $\{u_s\},\{v_s\}$ $u,v\in\mathcal{X}$, $\mathcal{D}^*_d(u_s,u,u)\longrightarrow 0$ and $\mathcal{D}^*_d(u_s,v_s,v_s)\longrightarrow 0\Longrightarrow \mathcal{D}^*_d(v_s,u,u)\longrightarrow 0$.

(W_5) Let $(\mathcal{X},\mathcal{D}^*_d)$ be complete $\mathcal{D}^*$-symmetric-sp. For $\{u_s\}$ in $\mathcal{X}$ we have $lim_{s,r\to\infty}\mathcal{D}^*_d(u_s,u_r,u_r)=0$ *iff* $lim_{s\to\infty}\mathcal{D}^*_d(u_s,u_{s+1},u_{s+1})=0$.

Definition2.6: [13] Presume $(\mathcal{X},\mathcal{D}^*_d)$ IS $\mathcal{D}^*$-symmetric-sp, so:

(i) $(\mathcal{X},\mathcal{D}^*_d)$ is $\mathcal{D}^*_d$-complete if for each $\mathcal{D}^*_d$-Cauchy-sequence $\{u_s\}$, $\exists\, u\in\mathcal{X}$ with $lim_{s\to\infty}\mathcal{D}^*_d(u_s,u,u)=0$.

(ii) $\mathcal{F}:(\mathcal{X},\mathcal{D}^*_d)\longrightarrow(\mathcal{X},\mathcal{D}^*_d)$ is $\mathcal{D}^*_d$-continuous map if $\lim_{s\to\infty}\mathcal{D}^*_d(u_s,u,u)=0$ implies that $lim_{s\to\infty}\mathcal{D}^*_d(\mathcal{F}u_s,\mathcal{F}u,\mathcal{F}u)=0$.

Definition 2.7: [17] $\mathcal{T}:\mathcal{X}\to 2^{\mathcal{X}}$ is called multi-valued map. $u\in\mathcal{X}$ is fixed point of $\mathcal{T}$ if $u\in\mathcal{T}_u$.

Definition2.8: [18] Presume $\mathcal{X}=\emptyset$. Presume that $\mathcal{G}:\mathcal{X}\to\mathcal{X}$ & $\mathcal{T}:\mathcal{X}\to 2^{\mathcal{X}}$. If $c=\mathcal{G}u\in\mathcal{T}u$ for some $u\in\mathcal{X}$, so $u$is coincidence point for $\mathcal{G},\mathcal{F}$, with c point of coincidence for $\mathcal{G}$ & F.

Definition2.9: [18] $\mathcal{G}:\mathcal{X}\to\mathcal{X}$ & $\mathcal{T}:\mathcal{X}\to 2^{\mathcal{X}}$ are called weakly compatible if $\in\mathcal{T}u,\forall\, u\in\mathcal{X},\Longrightarrow\mathcal{G}\mathcal{T}u\subseteq\mathcal{T}\mathcal{G}u$.

Definition 2.10: [18] Two mappings $\mathcal{G}:\mathcal{X}\to\mathcal{X}$ & $\mathcal{T}:\mathcal{X}\to 2^{\mathcal{X}}$ are called occasionally weakly compatible (briefly. O.W.C) *iff* $\exists\, u\in\mathcal{X}$ (s. t) $\mathcal{G}u\in\mathcal{T}u$ & $\mathcal{G}\mathcal{T}u\subseteq\mathcal{T}\mathcal{G}u$.

Remark 2.11: [18] An (O.W.C) map is weakly compatible, but not vice-versa.

Remark 2.12: We will denote for the collection of all non-empty closed and bounded subsets of $\mathcal{D}^*$-symmetric-sp $(\mathcal{X},\mathcal{D}^*_d)$ via $\mathcal{CB}(\mathcal{X})$.

The next definitions analogies for definitions are found [7] in context $\mathcal{D}^*$-symmetric-sp.

Definition 2.13: Two mappings $\mathcal{T},\mathcal{P}:\mathcal{X}\to\mathcal{CB}(\mathcal{X})$ are called weak contraction if $\exists\; 0<\Omega_1<1$ (s. t),

$\mathcal{H}_{\mathcal{D}^*_d}(\mathcal{P}v,\mathcal{P}v,\mathcal{T}u)\le\Omega_1\mathcal{M}(v,v,u)$ & $\mathcal{H}_{\mathcal{D}^*_d}(\mathcal{P}v,\mathcal{T}u,\mathcal{T}u)\le\Omega_1\mathcal{M}(v,u,u),\forall u,v\in\mathcal{X}$, where $\mathcal{M}(v,v,u)=$

$$max\{\mathcal{D}^*_d(v,v,u),\mathcal{D}^*_d(\mathcal{T}u,\mathcal{T}u,u),\mathcal{D}^*_d(\mathcal{P}v,\mathcal{P}v,v),\mathcal{D}^*_d(\mathcal{P}v,\mathcal{P}v,u),\mathcal{D}^*_d(\mathcal{T}u,\mathcal{T}u,v)\}.$$

With, $\mathcal{M}(v,u,u)=$

$$max\{\mathcal{D}^*_d(v,u,u),\mathcal{D}^*_d(\mathcal{T}u,u,u),\mathcal{D}^*_d(\mathcal{P}v,v,v),\mathcal{D}^*_d(\mathcal{P}v,u,u),\mathcal{D}^*_d(\mathcal{T}u,v,v)\}.$$

Definition 2.14: A $\mathcal{T}:\mathcal{X}\to\mathcal{CB}(\mathcal{X})$ is called weak contraction if $\exists\; 0<\Omega_1<1$ (s. t),

$\mathcal{H}_{\mathcal{D}^*_d}(\mathcal{T}v,\mathcal{T}v,\mathcal{T}u)\le\Omega_1\mathcal{N}(v,v,u)$ & $\mathcal{H}_{\mathcal{D}^*_d}(\mathcal{T}v,\mathcal{T}u,\mathcal{T}u)\le\Omega_1\mathcal{N}(v,u,u),\forall\, u,v\in\mathcal{X}$, Where, $\mathcal{N}(v,v,u)=$

$$max\{\mathcal{D}^*_d(v,v,u),\mathcal{D}^*_d(\mathcal{T}u,\mathcal{T}u,u),\mathcal{D}^*_d(\mathcal{T}v,\mathcal{T}v,v),\mathcal{D}^*_d(\mathcal{T}v,\mathcal{T}v,u),\mathcal{D}^*_d(\mathcal{T}u,\mathcal{T}u,v)\}.$$

With, $\mathcal{M}(v,u,u)=$

$$max\{\mathcal{D}^*_d(v,u,u),\mathcal{D}^*_d(\mathcal{T}u,u,u),\mathcal{D}^*_d(\mathcal{T}v,v,v),\mathcal{D}^*_d(\mathcal{T}v,u,u),\mathcal{D}^*_d(\mathcal{T}u,v,v)\}.$$

Where, $\mathcal{H}_{\mathcal{D}^*_d}$ symbolizes the Hausdorff $\mathcal{D}^*$-symmetric on $\mathcal{CB}(\mathcal{X})$ generated via $\mathcal{D}^*_d$, i.e.

$$\mathcal{H}_{\mathcal{D}^*_d}(\mathcal{B},\mathcal{B},\mathcal{A})=max\{Sup_{u\in\mathcal{A}}\mathcal{D}^*_d(\mathcal{B},\mathcal{B},u),Sup_{v\in\mathcal{B}}\mathcal{D}^*_d(\mathcal{A},\mathcal{A},v)\},\forall\,\mathcal{B},\mathcal{A}\in\mathcal{CB}(\mathcal{X}).$$

Main Results of Fixed Point Theorems for Multi-Valued Contraction Mappings in $\mathcal{D}^*$-Symmetric Space

Our stimulus of introduce this section, is to study and verify the existence and uniqueness of common fixed points for multi-valued contraction maps, as well the common fixed point theorems for (O.W.C) maps have been verified in $\mathcal{D}^*$-symmetric-sp. Our first main outcome is expansion of Theorem 2.2 [4] of $\mathcal{D}^*$-symmetric-sp.

Theorem.3.1: Let $(\mathcal{X},\mathcal{D}_d^*)$ be complete $\mathcal{D}_d^*$-symmetric-sp satisfies $(W_4)\&(W_5)$ where
$(\boldsymbol{i})$ $\mathcal{F}:\mathcal{X}\to\mathbb{R}$ described via $\mathcal{F}(u)=\mathcal{D}_d^*(\mathcal{T}u,\mathcal{T}u,u), u\in\mathcal{X}$, is semi-continuous;
$(\boldsymbol{ii})$ $\mathcal{T}:\mathcal{X}\to\mathcal{C}(\mathcal{X})$ multi-valued map where:

$$\mathcal{W}(\mathcal{T}v,\mathcal{T}v,\mathcal{T}u)\le h\mathcal{D}_d^*(v,v,u), h\in[0,1) \text{ ………………} \mathbf{(3.1)}$$

$\forall\, u,v\in\mathcal{X}$. in that case, $\exists\, p\in\mathcal{X}$ (s. t) $p\in\mathcal{T}p$

Proof: Presume $u_1\in\mathcal{X}$ & $\Omega_1\in(h,1)$. Because, $\mathcal{T}u_1$ is non-empty, $\exists\, u_2\in\mathcal{T}u_1$ (s. t), $\mathcal{D}_d^*(u_2,u_2,u_1)=\mathcal{D}_d^*(\mathcal{T}u_1,\mathcal{T}u_1,u_1)$. In observation of Eq-(3.1) get,

$$\begin{aligned}\mathcal{D}_d^*(\mathcal{T}u_2,\mathcal{T}u_2,u_2)&\le\mathcal{W}(\mathcal{T}u_2,\mathcal{T}u_2,\mathcal{T}u_1)\\&\le h\mathcal{D}_d^*(u_2,u_2,u_1)\\&<\Omega_1\mathcal{D}_d^*(u_2,u_2,u_1).\end{aligned}$$

In the same way, $\exists\, u_3\in\mathcal{T}u_2$ where $\mathcal{D}_d^*(u_3,u_3,u_2)=\mathcal{D}_d^*(\mathcal{T}u_2,\mathcal{T}u_2,u_2)$ with

$$\begin{aligned}\mathcal{D}_d^*(\mathcal{T}u_3,\mathcal{T}u_3,u_3)&\le\mathcal{W}\mathcal{D}_d^*(\mathcal{T}u_3,\mathcal{T}u_3,\mathcal{T}u_2)\\&\le h\mathcal{D}_d^*(u_3,u_3,u_2)\\&<\Omega_1\mathcal{D}_d^*(u_3,u_3,u_2).\end{aligned}$$

As a result, obtain a $\{u_s\}_{\ge1}^{\infty}$ in $\mathcal{X}$ satisfying $u_{s+1}\in\mathcal{T}u_s$ where

$$\mathcal{D}_d^*(u_{s+1},u_{s+1},u_s)=\mathcal{D}_d^*(\mathcal{T}u_s,\mathcal{T}u_s,u_s)$$

And have

$$\begin{aligned}\mathcal{D}_d^*(\mathcal{T}u_s,\mathcal{T}u_s,u_s)&\le\mathcal{W}\mathcal{D}_d^*(\mathcal{T}u_s,\mathcal{T}u_s,\mathcal{T}u_{s-1})\\&\le h\mathcal{D}_d^*(u_s,u_s,u_{s-1})\\&<\Omega_1\mathcal{D}_d^*(u_s,u_s,u_{s-1})\\&<(\Omega_1)^{s-1}\mathcal{D}_d^*(u_2,u_2,u_1).\end{aligned}$$

This implies $lim_{s\to\infty}\mathcal{D}_d^*(\mathcal{T}u_s,\mathcal{T}u_s,u_s)=0$. As, $\mathcal{D}_d^*(u_{s+1},u_{s+1},u_s)=\mathcal{D}_d^*(\mathcal{T}u_s,\mathcal{T}u_s,u_s)$, in that case $lim_{s\to\infty}\mathcal{D}_d^*(u_{s+1},u_{s+1},u_s)=0$,and with $(\boldsymbol{W_5})$, get $\{u_s\}$ is Cauchy sequence.
Consequently, $\exists\, p\in\mathcal{X}$ (s. t) $u_s\to p$. Additionally via $(\boldsymbol{i})$ wherever $\mathcal{F}(p)=\mathcal{D}_d^*(\mathcal{T}p,\mathcal{T}p,p)$ get

$$0\le\mathcal{D}_d^*(\mathcal{T}p,\mathcal{T}p,p)\le\lim_{s\to\infty}\mathcal{D}_d^*(\mathcal{T}u_s,\mathcal{T}u_s,u_s)=0.$$

Because, $\mathcal{T}$ is closed and $(\mathcal{X},\mathcal{D}_d^*)$ is complete, so $p\in\mathcal{T}p$. Consequently p is fixed point of $\mathcal{T}$.

Corollary 3.2: Suppose $(\mathcal{X},\mathcal{D}_d^*)$ is bounded and complete $\mathcal{D}_d^*$-symmetric-sp which satisfies (W_4) and $\mathcal{T}:\mathcal{X}\to\mathcal{C}(\mathcal{X})$ is a multi-valued map where:

$$\mathcal{D}_d^*(\mathcal{T}v,\mathcal{T}v,\mathcal{T}u)\le h\mathcal{D}_d^*(v,v,u)\text{………………} \mathbf{(3.2)}$$

$\forall\, u,v\in\mathcal{X}$ and $h\in[0,1)$. Then, $\exists\, p\in\mathcal{X}$ where $p\in\mathcal{T}p$.

Proof: Its follows immediately via Theorem-3.1.

Next Lemma needed to next Theorem, indicate the authors to [19] to its evidence.

Lemma3.3: Let$\mathcal{A},\mathcal{B}\in\mathcal{CB}(\mathcal{X})$ & $z\in\mathcal{A}$, so $\forall\,\varepsilon>0,\exists\, y\in\mathcal{B}$, where

$$\mathcal{D}_d^*(y,y,z)\le\mathcal{H}_{\mathcal{D}_d^*}(\mathcal{B},\mathcal{B},\mathcal{A})+\varepsilon.$$

Next Theorem is an analogy outcome of Theorem 3.1, in [20] in $\mathcal{D}^*$-symmetric-sp.

Theorem 3.4: Let $\mathcal{T}$ & $\mathcal{P}$ be multi-valued weak-contraction map of $\mathcal{D}^*$-symmetric-sp$(\mathcal{X},\mathcal{D}_d^*)$ where the pair $\{\mathcal{T},\mathcal{P}\}$ is (O.W.C). If $\forall\, u, v \in \mathcal{X}$

$$\mathcal{H}_{\mathcal{D}_d^*}(\mathcal{P}v,\mathcal{P}v,\mathcal{T}u) \le \Omega_1\mathcal{M}(v,v,u) \quad \ldots\ldots\ldots\ldots\ldots\ldots \textbf{(3.3)}$$

And

$$\mathcal{H}_{\mathcal{D}_d^*}(\mathcal{P}v,\mathcal{T}u,\mathcal{T}u) \le \Omega_1\mathcal{M}(v,u,u) \quad \ldots\ldots\ldots\ldots\ldots\ldots \textbf{(3.4)}$$

Where, $0 \le \Omega_1 < 1$ and $\Omega_2 = \Omega_1 + \varepsilon < 1$. So, $\exists\, u \in \mathcal{X}$ such that $u \in \mathcal{T}u$ & $u \in \mathcal{P}u$ ($\mathcal{T}$ & $\mathcal{P}$ have common fixed point of $\mathcal{X}$).

Proof: Because $\{\mathcal{T},\mathcal{P}\}$ are (O.W.C), in that case $\exists\, u \in \mathcal{X}$ (s. t) $\mathcal{T}u \in \mathcal{P}u$. Presume $\exists\, u_1, u_2 \in \mathcal{X}$ (s. t) $u_1 \in \mathcal{T}u_1$ & $u_2 \in \mathcal{P}u_2$ with $\mathcal{T}\mathcal{P}u_2 \subseteq \mathcal{P}\mathcal{T}u_1$, so utilizing Eq-(3.3) and Lemma **3.3,** we illustrate $u_1 = u_2$. On the contrary presume $u_1 \ne u_2$ and get,

$$\begin{aligned}\mathcal{D}_d^*(u_2,u_2,u_1) &\le \mathcal{H}_{\mathcal{D}_d^*}(\mathcal{P}u_2,\mathcal{P}u_2,\mathcal{T}u_1) + \varepsilon\mathcal{M}(u_2,u_2,u_1)\\ &\le \Omega_1\mathcal{M}(u_2,u_2,u_1) + \varepsilon\mathcal{M}(u_2,u_2,u_1)\\ &= \Omega_2\mathcal{M}(u_2,u_2,u_1)\end{aligned}$$

$$= \Omega_2 max\left\{\begin{matrix}\mathcal{D}_d^*(u_2,u_2,u_1),\mathcal{D}_d^*(\mathcal{T}u_1,\mathcal{T}u_1,u_1),\mathcal{D}_d^*(\mathcal{P}u_2,\mathcal{P}u_2,u_2)\\ ,\mathcal{D}_d^*(\mathcal{P}u_2,\mathcal{P}u_2,u_1),\mathcal{D}_d^*(\mathcal{T}u_1,\mathcal{T}u_1,u_2)\end{matrix}\right\}$$

$$\le \Omega_2 max\left\{\begin{matrix}\mathcal{D}_d^*(u_2,u_2,u_1),\mathcal{D}_d^*(u_1,u_1,u_1),\mathcal{D}_d^*(u_2,u_2,u_2),\\ \mathcal{D}_d^*(u_2,u_2,u_1),\mathcal{D}_d^*(u_1,u_1,u_2)\end{matrix}\right\}$$

$$\le \Omega_2 max\{\mathcal{D}_d^*(u_2,u_2,u_1),\mathcal{D}_d^*(u_1,u_1,u_2)\}.$$

Next, discuss two cases as shown below:

Case-(***i***)**:** Presume $max\{\mathcal{D}_d^*(u_2,u_2,u_1),\mathcal{D}_d^*(u_1,u_1,u_2)\} = \mathcal{D}_d^*(u_2,u_2,u_1)$. In that case $\mathcal{D}_d^*(u_2,u_2,u_1) \le \Omega_2\mathcal{D}_d^*(u_2,u_2,u_1)$.

Case-(***ii***)**:** If $max\{\mathcal{D}_d^*(u_2,u_2,u_1),\mathcal{D}_d^*(u_1,u_1,u_2)\} = \mathcal{D}_d^*(u_1,u_1,u_2)$. In that case

$$\mathcal{D}_d^*(u_2,u_2,u_1) \le \Omega_2\mathcal{D}_d^*(u_1,u_1,u_2) \quad \ldots\ldots\ldots\ldots\ldots\ldots \textbf{(3.5)}$$

In the same method, utilizing Equation-(3.4), obtain

$$\mathcal{D}_d^*(u_2,u_1,u_1) \le \mathcal{H}_{\mathcal{D}_d^*}(\mathcal{P}u_2,\mathcal{T}u_1,\mathcal{T}u_1) + \varepsilon\mathcal{M}(u_2,u_1,u_1)$$

$$\le \Omega_1\mathcal{M}(u_2,u_1,u_1) + \varepsilon\mathcal{M}(u_2,u_1,u_1) = \Omega_2\mathcal{M}(u_2,u_1,u_1) =$$

$$\Omega_2 max\{\mathcal{D}_d^*(u_2,u_1,u_1),\mathcal{D}_d^*(\mathcal{T}u_1,u_1,u_1),\mathcal{D}_d^*(\mathcal{P}u_2,u_2,u_2),\mathcal{D}_d^*(\mathcal{P}u_2,u_1,u_1),\mathcal{D}_d^*(\mathcal{T}u_1,u_2,u_2)\}$$

$$\le \Omega_2 max\{\mathcal{D}_d^*(u_2,u_1,u_1),\mathcal{D}_d^*(u_1,u_1,u_1),\mathcal{D}_d^*(u_2,u_2,u_2),\mathcal{D}_d^*(u_2,u_1,u_1),\mathcal{D}_d^*(u_1,u_2,u_2)\}$$

$$\le \Omega_2 max\{\mathcal{D}_d^*(u_2,u_1,u_1),\mathcal{D}_d^*(u_1,u_2,u_2)\}.$$

Next, discuss two cases as shown below:

Case-(***i***)**:** Assume $max\{\mathcal{D}_d^*(u_2,u_1,u_1),\mathcal{D}_d^*(u_1,u_2,u_2)\} = \mathcal{D}_d^*(u_2,u_1,u_1)$. In that case $\mathcal{D}_d^*(u_2,u_1,u_1) \le \Omega_2\mathcal{D}_d^*(u_2,u_1,u_1)$

Case-(***ii***)**:** If $max\{\mathcal{D}_d^*(u_2,u_1,u_1),\mathcal{D}_d^*(u_1,u_2,u_2)\} = \mathcal{D}_d^*(u_1,u_2,u_2)$. In that case

$$\mathcal{D}_d^*(u_2,u_1,u_1) \le \Omega_2\mathcal{D}_d^*(u_1,u_2,u_2)\ldots\ldots\ldots\ldots\ldots\ldots \textbf{(3.6)}$$

Merging Eq-**(3.5)** & **(3.6)** with $\mathcal{D}_d^*(4)$ produces $\mathcal{D}_d^*(u_2,u_2,u_1) \le \Omega_2\mathcal{D}_d^*(u_2,u_2,u_1)$. As, $\Omega_2 < 1$, so $u_1 = u_2$. This implies $u_1 \in \mathcal{T}u_1$ and $u_1 \in \mathcal{P}u_1$. For this reason $\mathcal{T}$ & $\mathcal{P}$ have common fixed point.

Remark 3.5: The next corollary yields from **Theorem 3.4,** when $\mathcal{T} = \mathcal{P}$.

Corollary 3.6: Assume $(\mathcal{X},\mathcal{D}_d^*)$ is complete $\mathcal{D}^*$-symmetric-sp, $\mathcal{T}: \mathcal{X} \to \mathcal{CB}(\mathcal{X})$ is multi-valued weak-contraction map where $\forall\, u, v \in \mathcal{X}$

$$\mathcal{H}_{\mathcal{D}_d^*}(\mathcal{T}v,\mathcal{T}v,\mathcal{T}u) \le \Omega_1\mathcal{N}(v,vu)\ldots\ldots\ldots\ldots\ldots\ldots\ldots \textbf{(3.7)}$$

Where, $0 \le \Omega_1 < 1$. Then, $\exists\, u \in \mathcal{X}$ (s. t) $u \in \mathcal{T}u$.

Example 3.7: Presume $\mathcal{X} = [0,1]$ is given with $\mathcal{D}^*$-symmetric, and $\mathcal{P},\mathcal{T}: \mathcal{X} \to \mathcal{CB}(\mathcal{X})$ are maps described via $\mathcal{T}u = \left[0,\frac{1}{6}u\right]$ and $\mathcal{P}v = \left[\frac{1}{6}v\right]$.

$$\mathcal{H}_{\mathcal{D}_d^*}(\mathcal{P}v, \mathcal{P}v, \mathcal{T}u) = max\left\{\left(\frac{1}{6}v\right)^2, \left(\frac{1}{6}v - \frac{1}{6}u\right)^2\right\}$$
$$\leq \frac{1}{3}max\left\{\left(v - \frac{v}{6}\right)^2, (v-u)^2\right\}$$
$$\leq \frac{1}{3}max\{\mathcal{D}_d^*(\mathcal{P}v, \mathcal{P}v, v), \mathcal{D}_d^*(v, v, u)\} \leq \frac{1}{3}\mathcal{M}(v, v, u).$$

Consequently, $\mathcal{T}$ & $\mathcal{P}$ have common fixed point is $(u = 0)$.

Conclusion

The idea of fixed-point theory provides an efficient in many fields of mathematics and other sciences; also this theory for multi-valued contraction mappings has acquired formidable implementation in venous sciences such as economics, optimization, control theory and deferential equations . So, in the present article numerous of "fixed point theorems" multi-valued contraction maps are verified. Additionally, existence of common fixed-point outcomes to (O.W.C) maps under influence multi-valued weak-contractive conditions in complete $\mathcal{D}^*$-symmetric-sp has been established. Our major outcomes in this article extend, and improvement of various outcomes in the literature. We anticipate that the discoveries in this article will aid scientists in enhancing the researches on extended symmetric spaces in order to elevate a general framework for their practical implementations in all advanced branches of pure and applied mathematics and other sciences .

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حول التطبيقات الانكماشية متعددة القيم ونظريات النقطة الثابتة في الفضاءات المتماثلة الكاملة

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الخلاصة:

الدافع الرئيسي من تقديم هذه المقالة هو لدراسة والتحقق من نظريات النقطة الثابتة المتنوعة للتطبيقات الانكماشية متعددة القيم، كما تم إثبات وجود ووحدانية النقاط الثابتة المشتركة لزوج من التطبيقات المتوافقة بشكل ضعيف عرضيًا تحت تأثير مجموعة من الشروط العامة للانكماش الضعيف متعدد القيم في سياق فضاء متماثل موسع جديد يُعرف بفضاء*D- المتناظر، والذي يُعد امتدادًا لفضائي *D-المتري وG- المتناظر؛ وتشمل النتائج الرئيسية امتدادات وتعميمات لنتائج عديدة موجودة في الأدبيات ، مع تقديم عدة أمثلة مناسبة تدعم النتائج الرئيسية التي تم الحصول عليها.