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Solving Numerical Solution of the Generalised Fisher Equation Using the Explicit Euler and the Crank-Nicholson Methods

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Abstract

This study presents a numerical solution to the generalized Fisher equation (GFE) using two finite- difference numerical methods: the explicit method and the Crank-Nicholson method. The convergence for each method was analysed theoretically and experimentally, while examining the effect of the parameters δ on the system dynamics. The results showed that increasing δ leads to faster solution evolution and higher solution values. The explicit method was also computationally more efficient, while the Crank-Nicholson method outperformed the GFE in accuracy and stability. Using small time and spatial steps significantly improved the accuracy of the results. The results were validated through various numerical examples using MATLAB R2022a, confirming the effectiveness of both methods in modeling the complex phenomena described by the equation.

Introduction

The Fisher equation (FE) is not only mathematical formula; it is a powerful tool for understanding the interactions between temporal and spatial variations in natural and artificial systems [1]. It remains one of fundamental models that contains to motivate scientific and mathematical research, making it key topic in applied mathematics modelling [2]. The generalized Fisher equation (GFE) is a fundamental model in study of gene diffusion, chemical reactions, and heat transfer, making improving its numerical solution of great value in these fields [3].

FE, introduced by Ronald in 1937, has been generalized. The GFE is development of the basic equation and has been extended in various ways to model more complex phenomena in fields such as biology, and physics. This equation expresses the interaction between diffusion and non-linear growth, making it powerful tool for studying processes involving propagating waves [2].

The generalised Fisher equation is a second order parabolic semilinear partial differential equation:

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = \beta u\left(1 - u^\delta\right), \quad (1)$$

At $\delta > 0$, thus, when $\delta = 0$, reduces to traditional FE. Which also known as Fisher-KPP equation or KPP-Fisher equation or Fisher-Kolomogrov-Petrovsky-Piskunov equation or KPP equation

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = \beta u(1 - u). \quad (2)$$

The generalised Burger-Fisher equation (3) is well-known model of convection-diffusion-reaction equations

$$\frac{\partial u}{\partial t} + \alpha u^\delta \frac{\partial u}{\partial x} - D\frac{\partial^2 u}{\partial x^2} = \beta u\left(1 - u^\delta\right). \quad (3)$$

thus, when $\beta = 0$, reduces to generalised Burger's equation.

$$\frac{\partial u}{\partial t} + \alpha u^\delta \frac{\partial u}{\partial x} - D\frac{\partial^2 u}{\partial x^2} = 0. \quad (4)$$

when $\delta = 1$ in (4), we have the famous Burger's equation

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} - D\frac{\partial^2 u}{\partial x^2} = 0. \quad (5)$$

Many studies presented related to GFE, as analytical and numerical solution of GFE are discussed using symmetry methods and mathematical analysis to study the properties of propagating waves [1]. Kundu studies traveling wave solutions, the Painlevé analysis of the generalised Fisher equation, and the Lotka–Volterra diffusion model [2]. Zhong presents the higher-order Haar wave diagram with the Runge–Kutta method to solve the generalised Burgers–Fisher and Burgers–Huxley equations [3]. Exact traveling wave solutions of the generalised Fisher equation are also explored using analytical techniques, with a focus on physical and biological applications [4]. Drábek and M. Zahradníková studied propagating waves in the generalised Fisher–Kolmogorov equation with discontinuous density-dependent diffusion, highlighting the effect of non-uniform variations [5]. Abd-Elhameed et al. used a new linear formulation based on non-symmetric Jacobi polynomials for numerical treatment of the nonlinear Fisher equation, which improves numerical solution techniques [6], also they provided a generalisation of Djeumen Tchaho formulas for analytical solutions and developed new wavelet solutions for the generalised Fisher equation, which enhances the understanding of its dynamic behavior [7], and Yadav Singh developed the Elzaki variational iteration method for solving the generalised time-fraction Burgers–Fisher equation in porous media flow modelling, which contributes to nonlinear transport analysis [8]. The authors in [11-19] used finite difference methods for solving Fisher and Huxley equations and they studied the stability of the Burger's equation.

The importance of this research lies in providing a numerical solution to GFE and analyzing the convergence of the explicit and Crank methods, which contributes to understanding their performance and determining the optimal conditions for their use. It also studies the effect of the parameter δ on the dynamics of the solution, providing valuable insights for practical applications.

In this paper, we consider solving the generalised Fisher equation numerically using the explicit (forward) Euler method and the Crank-Nicholson method. Also, the convergence (error) analysis of both schemes is examined. The paper is organised as following: section 1 is a general introduction and literature review about the generalised Fisher equation. In section

2, we present the numerical schemes and the convergence analysis is investigated in section 3. The conclusions are given in section 4.

The Explicit Euler and Crank Nicholson Schemes

The explicit method is a finite difference method used to numerically solve partial differential equations, particularly, the time-dependent problems such as the heat equation and the diffusion equation. It relies on estimating the value of a function at a future time step using its values at the current time step, without the need to solve a system of linear equations [7]. The explicit method is easy to implement, does not require solving systems of linear equations, and is highly computationally efficient when using small time steps [9]. To derive the explicit method of the generalised Fisher equation, the time derivative $\frac{\partial u}{\partial t}$ at t_j is approximated by the forward difference approximation

$$\frac{\partial u}{\partial t}=\frac{u_{i,j+1}-u_{i,j}}{\Delta t}, \quad (6)$$

and the spatial second derivative is approximate using the central difference approximation

$$\frac{\partial^2 u}{\partial x^2}=\frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{\Delta x^2}. \quad (7)$$

Then substituting the finite difference approximations (6) and (7) in the generalised Fisher equation (1) which results in:

$$\frac{u_{i,j+1}-u_{i,j}}{\Delta t}-D\frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{\Delta x^2}=\beta u_{i,j}\left(1-\left(u_{i,j}\right)^{\delta}\right). \quad (8)$$

Rearranging the terms in (8) and solving it for $u_{i,j+1}$ to have:

$$u_{i,j+1}=u_{i,j}+D\frac{\Delta t}{\Delta x^2}\left(u_{i-1,n}-2u_{i,j}+u_{i+1,j}\right)+\Delta t\beta u_{i,n}\left(1-\left(u_{i,j}\right)^{\delta}\right). \quad (9)$$

Using $r=D\frac{\Delta t}{\Delta x^2}$, where r represents the parameter which determines the stability of the method, then we obtain the explicit scheme of the generalised Fisher equation

$$u_{i,j+1}=(1-2r)u_{i,j}+r\left(u_{i-1,j}+u_{i+1,j}\right)+\Delta t\beta u_{i,j}\left(1-\left(u_{i,j}\right)^{\delta}\right). \quad (10)$$

The Crank-Nicholson method is a semi-implicit method for solving partial differential equations. It is an improvement over the explicit and implicit methods. It averages the explicit and implicit methods to achieve better numerical stability. The Crank-Nicholson method is not as strict as the explicit method in time stability and is more accurate than the explicit method, as it is a second-order method in time and space, stable and not susceptible to numerical instability even at larger time steps. Its drawbacks include the requirement to solve a system of linear equations at each time step, which increases computational cost compared to the explicit method and can produce numerical oscillations in the solutions if the time and space steps are not well controlled [10]. The time derivative $\frac{\partial u}{\partial t}$ at $t_{j+1/2}$ is approximated by the central difference approximation

$$\frac{\partial u}{\partial t}=\frac{u_{i,j+1}-u_{i,j}}{\Delta t}, \quad (11)$$

and the spatial second derivative is approximate using the average of the central difference approximations at times t_j and t_{j+1}

$$\frac{\partial^2 u}{\partial x^2}=\frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{2\Delta x^2}+\frac{u_{i-1,j+1}-2u_{i,j+1}+u_{i+1,j+1}}{2\Delta x^2} \quad (12)$$

Then substituting the finite difference approximations (11) and (12) in the generalised Fisher equation (1) and rearranging the terms, we have the Crank-Nicholson scheme for the generalised Fisher equation.

$$-ru_{i-1,j+1}+(2+2r)u_{i,j+1}-ru_{i+1,j+1}=ru_{i-1,j}+(2-2r)u_{i,j}+ru_{i+1,j}+\Delta t\beta\left(u_{i,j}\right)\left(1-\left(u_{i,j}\right)^{\delta}\right). \quad (13)$$

The Convergence Analysis

In this section, we examine and study the error analysis of the numerical solutions of the generalised Fisher equation using the explicit Euler and Crank-Nicholson finite difference schemes.

The Convergence Analysis of the Explicit Scheme

The error is measured by the difference between the exact solution of the partial differential equation and the discrete numerical solution. Substituting the exact solution u(x, t) in (1), we have

$$T_{i,j} = \frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{k} - D\frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)}{h^2} - \beta u(x_i, t_j)\left(1 - \left(u(x_i, t_j)\right)^{\delta}\right). \quad (14)$$

Using Taylor's expansion of the function $u(x_i, t_{j+1})$ about t_j as follows

$$u(x_i, t_{j+1}) = u(x_i, t_j) + ku_t(x_i, t_j) + \frac{k^2}{2!}u_{tt}(x_i, t_j) + \frac{k^3}{3!}u_{ttt}(x_i, \rho_j), \quad (15)$$

where $\rho_j \in (t_j, t_{j+1})$. Also, using Taylor's expansion of the functions $u(x_{i+1}, t_j)$ and $u(x_{i-1}, t_j)$ about x_i, we find

$$u(x_{i+1}, t_j) = u(x_i, t_j) + hu_x(x_i, t_j) + \frac{h^2}{2!}u_{xx}(x_i, t_j) + \frac{h^3}{3!}u_{xxx}(x_i, t_j) + \frac{h^4}{4!}u_{xxxx}(\xi_i, t_j), \quad (16)$$

where $\xi_i \in (x_i, x_{i+1})$,

$$u(x_{i-1}, t_j) = u(x_i, t_j) - hu_x(x_i, t_j) + \frac{h^2}{2}u_{xx}(x_i, t_j) - \frac{h^3}{3!}u_{xxx}(x_i, t_j) + \frac{h^4}{4!}u_{xxxx}(\zeta_i, t_j), \quad (17)$$

where $\xi_i \in (x_{i-1}, x_i)$. Substituting (15) in (14), we have

$$\begin{aligned}\frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{k} &= \frac{u(x_i, t_{j+k}) - u(x_i, t_j)}{k}\\ &= \frac{u(x_i, t_j) + ku_t(x_i, t_j) + \frac{k^2}{2}u_{tt}(x_i, t_j) + \frac{k^3}{3!}u_{ttt}(x_i, \rho_j) - u(x_i, t_j)}{k}\\ &= \frac{ku_t(x_i, t_j) + \frac{k^2}{2}u_{tt}(x_i, t_j) + \frac{k^3}{3!}u_{ttt}(x_i, \rho_j)}{k}\\ &= u_t(x_i, t_j) + \frac{k}{2}u_{tt}(x_i, \rho_j) + \frac{k^2}{3!}u_{ttt}(x_i, \rho_j). \quad (18)\end{aligned}$$

In the same way, substituting (16) and (17) in the second term in (14), we obtain

$$\begin{aligned}&\frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)}{h^2}\\ &= \frac{u(x_i, t_j) + hu_x(x_i, t_j) + \frac{h^2}{2}u_{xx}(x_i, t_j) + \frac{h^3}{3!}u_{xxx}(x_i, t_j) + \frac{h^4}{4!}u_{xxxx}(\xi_i, t_j)}{h^2} - \frac{2u(x_i, t_j)}{h^2}\\ &+ \frac{u(x_i, t_j) - hu_x(x_i, t_j) + \frac{h^2}{2}u_{xx}(x_i, t_j) - \frac{h^3}{3!}u_{xxx}(x_i, t_j) + \frac{h^4}{4!}u_{xxxx}(\xi_i, t_j)}{h^2}\\ &= u_{xx}(x_i, t_j)\\ &+ \frac{h^4}{4!}\left(u_{xxxx}(\xi_i, t_j) + u_{xxxx}(\zeta_i, t_j)\right). \quad (19)\end{aligned}$$

By substituting (18) and (19) in (14), we have

$$T_i^n = \frac{k}{2}u_{tt}(x_i, t_j) - \frac{h^4}{4!}\left(u_{xxxx}(\xi_i, t_j) + u_{xxxx}(\zeta_i, t_j)\right) - \beta u(x_i, t_j)\left(1 - \left(u(x_i, t_j)\right)^{\delta}\right). \quad (20)$$

Continuing with the same approach, we have the equation in the form

$$T_i^n = \frac{k}{2}u_{tt}(x_i, t_j) - \frac{h}{24}u_{xx}(x_i, t_j) + O(k^2, h^4)$$

$$T_i^n = \lim_{k,h\to 0}\left\{\frac{k}{2}u_{tt}(x_i,t_j) - \frac{h}{24}u_{xx}(x_i,t_j) + O(k^2,h^4)\right\} = 0$$

We assume that u(x, t) satisfies the smooth transition condition of the above proof and that the diagram is the explicit method of the generalised Fisher equation. Convergent, and we have and satisfies the condition

$$\frac{(1+L_f)}{1+2R}E^n + k\left\{\frac{k}{2}u_{tt}(x_i,t_j) - \frac{h}{24}u_{xx}(x_i,t_j) + O(k^2,h^4)\right\}$$

Proof:

$$T_i^n = \frac{u(x_i,t_{j+1}) - u(x_i,t_j)}{k} - \frac{u(x_{i+1},t_{j+1}) - 2u(x_i,t_j) + u(x_{i-1},t_{j+1})}{h^2} - f(u) \qquad (19)$$

We rephrase the above equation as follows:

$$\frac{U_i^{n+1} - U_i^n}{k} - \frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{h^2} = f(U_i^n) \qquad (20)$$

Subtracting equations (20) and (19), we get:

$$\frac{u(x_i,t_{j+1}) - U_i^{n+1} - u(x_i,t_j) + U_i^n}{k} - \frac{u(x_{i+1},t_{j+1}) - U_{i+1}^{n+1} - 2u(x_i,t_{j+1}) + 2U_i^{n+1} + u(x_{i-1},t_{j+1}) - U_{i-1}^{n+1}}{h^2} - f(u(x,t)) + f(U_i^n)$$

We substitute $e_i^n = u(x_i,t_j) - U_i^n$ to get:

$$T_i^n = \frac{e_i^{n+1} - e_i^n}{k} - \frac{e_{i+1}^{n+1} - 2e_i^{n+1} + e_{i-1}^{n+1}}{h^2} - f(u(x,t)) + f(U_i^n)$$

Multiply the equation by k

$$kT_i^n + e_i^n - k[f(u(x,t)) + f(U_i^n)] = k\frac{e_{i+1}^{n+1} - 2e_i^{n+1} + e_{i-1}^{n+1}}{h^2} + e_i^{n+1}$$

Let $r = \frac{k}{h^2}$

$$kT_i^n + e_i^n - k[f(u(x,t)) + f(U_i^n)] = R(e_{i+1}^{n+1} + 2e_i^{n+1} + e_{i-1}^{n+1}) + e_i^{n+1}$$
$$kT_i^n + e_i^n - k[f(u(x,t)) + f(U_i^n)] = Re_{i+1}^{n+1} + Re_{i-1}^{n+1} + (1+2R)e_i^{n+1}$$

Let $E^n = \max\{e_i^{n+1}\}$

The maximum error at time step n, then taking the maximum error with respect to i on the right-hand side of, we arrive at applying the Lipschitz condition to the nonlinear limit, gives:

$$|f(u(x,t)) + f(U_i^n)| \le L_f|u(x,t) - U_i^n| \le L_f|e_i^n|$$

Leads to

$$e_i^{n+1} \le \frac{(1+L_f)}{1+2R}E^n + kT_i^n$$

Or taking the maximum with respect to i, and using the theorem above, we get

$$E^{n+1} \le \frac{(1+L_f)}{1+2R}E^n + k\left\{\frac{k}{2}u_{tt}(x_i,t_j) - \frac{h}{24}u_{xx}(x_i,t_j) + O(k^2,h^4)\right\}$$

The Convergence Analysis of the Crank-Nicholson Scheme

While the convergence condition proof for the Crank-Nicholson method can be derived by same way in explicit method. The reduction error is measured by the difference between the true partial differential equation and the discrete plot. Substituting the exact solution u(x, t) into the differential equation

$$T_i^n = \frac{U(x_i,t_{j+1}) - U(x_i,t_j)}{k} - \frac{U(x_{i+1},t_{j+1}) - 2U(x_i,t_{j+1}) + U(x_{i-1},t_{j+1})}{2h^2} - \frac{U(x_{i+1},t_j) - 2U(x_i,t_j) + U(x_{i-1},t_j)}{2h^2} - \beta\left[(U_i^n + U_i^{n+1})\left(1 - (U_i^n)^\delta - (U_i^{n+1})^\delta\right)\right] \qquad (21)$$

Using Taylor's expansion of the function $u(x_i,t_{j+1})$ about t_n as follows:

$$u(x_{i+1},t_j) = u(x_i,t_j) + ku_t(x_i,t_j) + \frac{k^2}{2}u_{tt}(x_i,t_j) + O(k^3) \qquad (22)$$

where $O(k^3)$ represents the derivatives with respect to u of order three and above. Again, using Taylor's expansion of the functions $u(x_{i+1}, t_j)$ and $u(x_{i-1}, t_j)$ about x_i, we find:

$$u(x_{i+1}, t_j) = u(x_i, t_j) + hu_x(x_i, t_j) + \frac{h^2}{2} u_{xx}(x_i, t_j) + O(h^3) \quad (23)$$

$$u(x_{i-1}, t_j) = u(x_i, t_j) - hu_x(x_i, t_j) + \frac{h^2}{2} u_{xx}(x_i, t_j) + O(h^3) \quad (24)$$

where $O(h^3)$ represents the derivatives with respect to x of the third order and above. We substitute the expansion of the above equations into equation (21), which is as follows:

$$\frac{u(x_i, t_j) + ku_t(x_i, t_j) + \frac{k^2}{2} u_{tt}(x_i, t_j) + O(k^3) - u(x_i, t_j)}{k}$$
$$- \frac{u(x_i, t_j) + ku_t(x_i, t_j) + \frac{k^2}{2} u_{tt}(x_i, t_j) + O(k^3)}{2h^2} - \frac{2u(x_i, t_{j+1})}{2h^2}$$
$$+ \frac{u(x_i, t_j) + ku_t(x_i, t_j) + \frac{k^2}{2} u_{tt}(x_i, t_j) + O(k^3)}{2h^2}$$
$$- \frac{u(x_i, t_j) + hu_x(x_i, t_j) + \frac{h^2}{2} u_{xx}(x_i, t_j) + O(h^3)}{2h^2} - \frac{2u(x_i, t_j)}{2h^2}$$
$$+ \frac{u(x_i, t_j) - hu_x(x_i, t_j) + \frac{h^2}{2} u_{xx}(x_i, t_j) + O(h^3)}{2h^2} - \beta[(u_i^n)(1 - (u_i^n)^\delta)]$$

Applying the same steps as in the explicit equation and simplifying the similar terms to get the following image:

$$T_i^n = u_t(x_i, t_j) + u_{xx}(x_i, t_j) + \beta(u_i^n)(1 - (u_i^n)^\delta) + O(k^2, h^2)$$

Or taking the maximum with respect to i, and using the same technique as in the explicit method, we get

$$E^{n+1} \leq \frac{(1 + L_f)}{1 + 2R} E^n + k\{u_t(x_i, t_j) + u_{xx}(x_i, t_j) + \beta(u_i^n)(1 - (u_i^n)^\delta) + O(k^2, h^2)\}.$$

Numerical Examples and Results

We provide several diverse instance of solving GFE for varying issue parameter values in this section.

Example 1

Take $x \in [0,1], t \in [0,0.2], \beta = \delta = 1$ by divided [0,1] at $n = 5$, the time interval [0,0.2] is split into $m = 10$ sub-intervals with step size $k = \Delta t = \frac{d-c}{m} = 0.02$, and the step size on x-axis $h = \frac{b-a}{n} = 0.2$.

Table 1: Example 1a: The explicit scheme results when $x \in [0, 1]$, $t \in [0, 0.2]$, $\beta = 1$, $h = 0.2$, $k = 0.02$ and $\delta = 1$.

	$x_1 = 0.2000$	$x_2 = 0.4000$	$x_3 = 0.6000$	$x_4 = 0.8000$	$x_5 = 1.0000$
$t_1 = 0.0200$	0.5344	0.5108	0.4869	0.4628	0.4859
$t_2 = 0.0400$	0.5392	0.5158	0.4919	0.4678	0.4879
$t_3 = 0.0600$	0.5440	0.5208	0.4969	0.4727	0.4899
$t_4 = 0.0800$	0.5487	0.5258	0.5019	0.4777	0.4919
$t_5 = 0.1000$	0.5535	0.5308	0.5069	0.4827	0.4939
$t_6 = 0.1200$	0.5582	0.5358	0.5119	0.4877	0.4959
$t_7 = 0.1400$	0.5629	0.5407	0.5169	0.4927	0.4979
$t_8 = 0.1600$	0.5676	0.5457	0.5219	0.4977	0.4999
$t_9 = 0.1800$	0.5722	0.5507	0.5269	0.5027	0.5019
$t_{10} = 0.2000$	0.5769	0.5556	0.5318	0.5077	0.5039

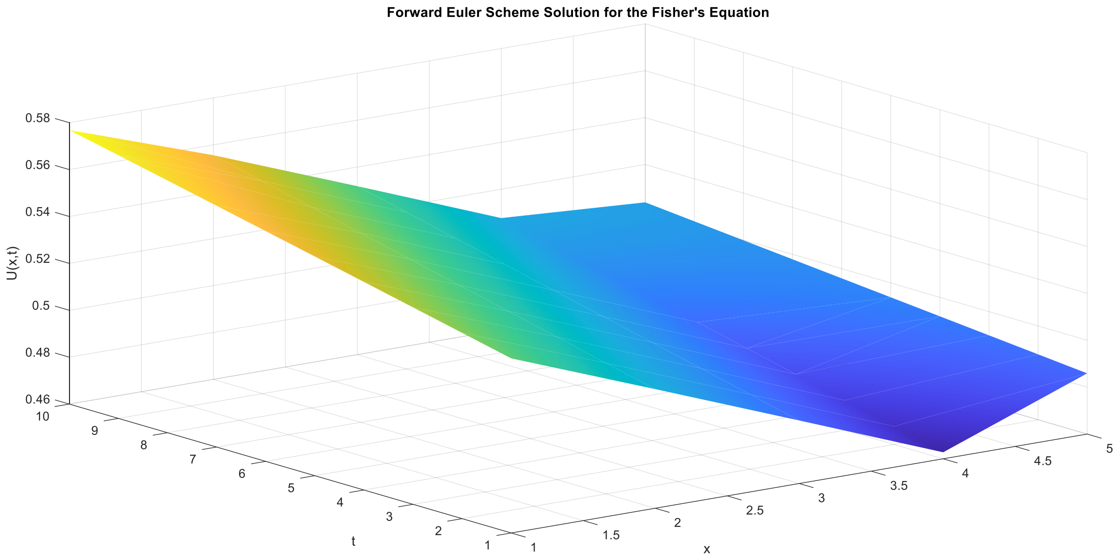

Fig.1: *Example 1a: The explicit scheme results when* $x \in [0,1], t \in [0,0.2], \beta = 1, h = 0.2, k = 0.02$ *and* $\delta = 1$.

Table 1 shows that the values of the solution gradually increase with increasing time (t) for each spatial point (x). The figure indicates that the solution evolves smoothly with time, with a slight increase in values at x = 1 compared to the rest of the points, which may indicate boundary effects.

Table 2: Example 1b: The Crank-Nicholson scheme results when $x \in [0,1], t \in [0,0.2], \beta = 1, h = 0.2, k = 0.02$ and $\delta = 1$.

	$x_1 = 0.200$	$x_2 = 0.400$	$x_3 = 0.600$	$x_4 = 0.800$	$x_5 = 1.000$
$t_1 = 0.0200$	0.5344	0.5108	0.4869	0.4628	0.4859
$t_2 = 0.0400$	0.5392	0.5158	0.4919	0.4678	0.4879
$t_3 = 0.0600$	0.5440	0.5208	0.4969	0.4727	0.4899
$t_4 = 0.0800$	0.5487	0.5258	0.5019	0.4777	0.4919
$t_5 = 0.1000$	0.5535	0.5308	0.5069	0.4827	0.4939
$t_6 = 0.1200$	0.5582	0.5358	0.5119	0.4877	0.4959
$t_7 = 0.1400$	0.5629	0.5407	0.5169	0.4927	0.4979
$t_8 = 0.1600$	0.5676	0.5457	0.5219	0.4977	0.4999
$t_9 = 0.1800$	0.5722	0.5507	0.5269	0.5027	0.5019
$t_{10} = 0.2000$	0.5769	0.5556	0.5318	0.5077	0.5039

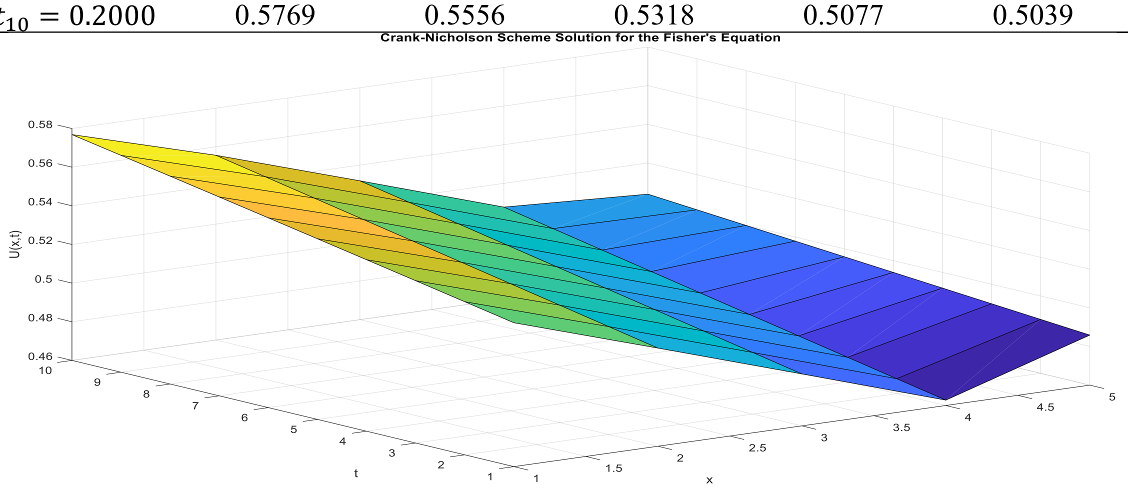

Fig.2: *Example 1b: The Crank Nicolson scheme results when* $x \in [0,1], t \in [0,0.2], \beta = 1, h = 0.2, k = 0.02$ *and* $\delta = 1$.

Table 2 shows a gradual increase in the solution values with increasing time (t) for each spatial point (x). Figure 2 indicates that the solution evolves smoothly with time, with a slight increase in values at x = 1 compared to the rest of the points, which may indicate boundary effects. The results are almost identical to the explicit scheme, indicating the stability and accuracy of both schemes for these parameters. The figure shows behavior similar to the solution, with gradually increasing values over time.

Example 2

Take $x \in [0,1], t \in [0,0.2], \beta = 1, \delta = 2$ by divided $[0,1]$ at $n = 5$, the time interval $[0,0.2]$ is split into $m = 10$ sub-intervals with step size $k0.02$, and the step size on x-axis $h = 0.2$.

Table 3: Example 2a: The explicit scheme results when $x \in [0,1], t \in [0,0.2]$, $\beta = 1, h = 0.2, k = 0.02$ and $\delta = 2$.

	$x_1 = 0.200$	$x_2 = 0.400$	$x_3 = 0.600$	$x_4 = 0.800$	$x_5 = 1.000$
$t_1 = 0.0200$	0.7311	0.7024	0.6720	0.6401	0.6792
$t_2 = 0.0400$	0.7369	0.7095	0.6794	0.6476	0.6815
$t_3 = 0.0600$	0.7427	0.7165	0.6867	0.6552	0.6838
$t_4 = 0.0800$	0.7484	0.7235	0.6939	0.6626	0.6861
$t_5 = 0.1000$	0.7540	0.7304	0.7011	0.6701	0.6884
$t_6 = 0.1200$	0.7595	0.7372	0.7083	0.6775	0.6906
$t_7 = 0.1400$	0.7649	0.7439	0.7153	0.6848	0.6929
$t_8 = 0.1600$	0.7703	0.7506	0.7223	0.6921	0.6951
$t_9 = 0.1800$	0.7756	0.7571	0.7292	0.6993	0.6974
$t_{10} = 0.2000$	0.7807	0.7636	0.7360	0.7064	0.6996

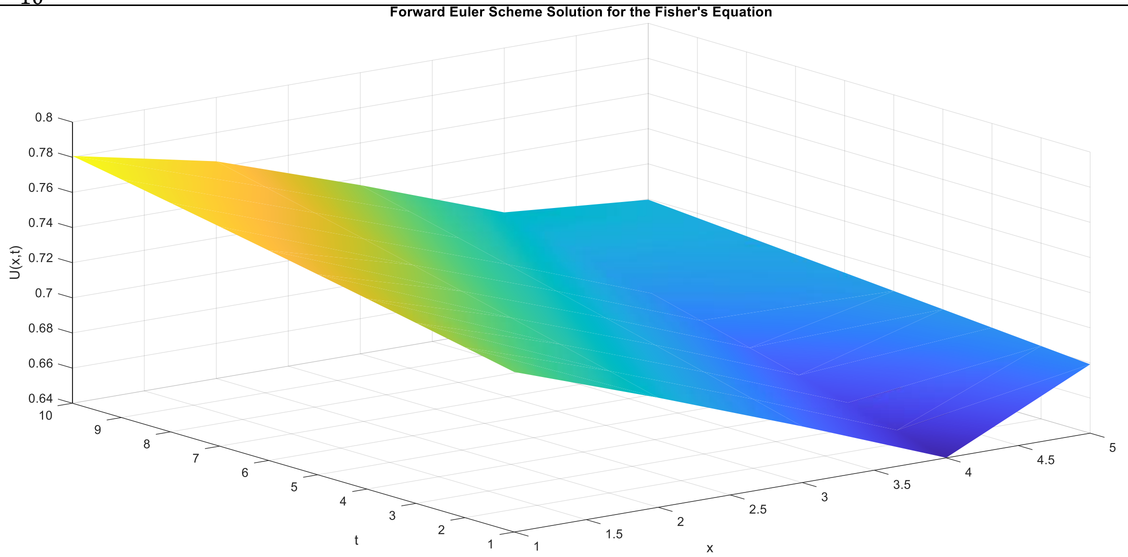

Fig.3: *Example 2a: The explicit scheme results when* $x \in [0,1], t \in [0,0.2], \beta = 1, h = 0.2, k = 0.02$ *and* $\delta = 2$.

From Table 3, it appears that the values are higher compared to Example 1 due to the increase in δ, indicating that δ increases the rate of change of the solution. The figure shows a steeper curve, reflecting the effect of δ on the dynamics of the equation.

Table 4: Example 2b: The Crank-Nicholson scheme results when $x \in [0,1], t \in [0,0.2], \beta = 1, h = 0.2, k = 0.02$ and $\delta = 2$.

	$x_1 = 0.200$	$x_2 = 0.400$	$x_3 = 0.600$	$x_4 = 0.800$	$x_5 = 1.000$
$t_1 = 0.0200$	0.7311	0.7024	0.6720	0.6401	0.6792
$t_2 = 0.0400$	0.7369	0.7095	0.6794	0.6476	0.6815
$t_3 = 0.0600$	0.7427	0.7165	0.6867	0.6552	0.6838
$t_4 = 0.0800$	0.7484	0.7235	0.6939	0.6626	0.6861
$t_5 = 0.1000$	0.7540	0.7304	0.7011	0.6701	0.6884
$t_6 = 0.1200$	0.7595	0.7372	0.7083	0.6775	0.6906
$t_7 = 0.1400$	0.7649	0.7439	0.7153	0.6848	0.6929
$t_8 = 0.1600$	0.7703	0.7506	0.7223	0.6921	0.6951
$t_9 = 0.1800$	0.7756	0.7571	0.7292	0.6993	0.6974
$t_{10} = 0.2000$	0.7807	0.7636	0.7360	0.7064	0.6996

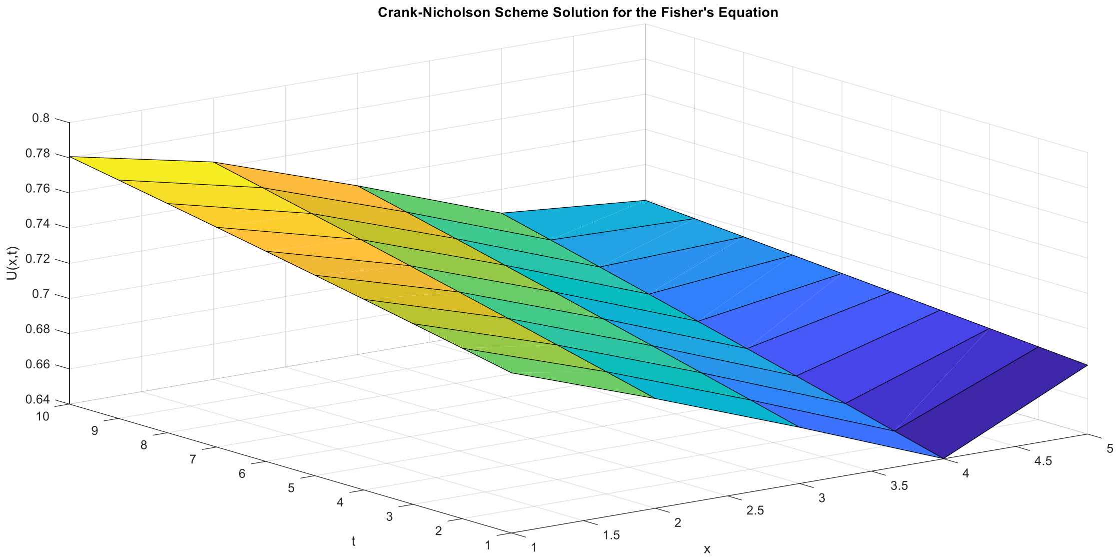

Fig.4: *Example 2b: The Crank-Nicholson scheme results when* $x \in [0,1], t \in [0,0.2], \beta = 1, h = 0.2, k = 0.02$ *and* $\delta = 2$.

From Table 4 it is shown that the results values are consistent with the explicit scheme, which confirms the accuracy of the two schemes.

Example 3

Take $x \in [0,1], t \in [0,0.2], \beta =, \delta = 3$ by divided $[0,1]$ at $n = 5$, the time interval $[0,0.2]$ is split into $m = 10$ sub-intervals with step size $k = 0.02$, and the step size on x-axis $h = 0.2$.

Table 5: Example 3a: The explicit scheme results when $x \in [0,1], t \in [0,0.2]$, $\beta = 1, h = 0.2, k = 0.02$ and $\delta = 3$.

	$x_1 = 0.200$	$x_2 = 0.400$	$x_3 = 0.600$	$x_4 = 0.800$	$x_5 = 1.000$
$t_1 = 0.0200$	0.8115	0.7825	0.7506	0.7162	0.7643
$t_2 = 0.0400$	0.8176	0.7906	0.7593	0.7252	0.7665
$t_3 = 0.0600$	0.8235	0.7986	0.7678	0.7342	0.7688
$t_4 = 0.0800$	0.8292	0.8065	0.7762	0.7431	0.7710
$t_5 = 0.1000$	0.8348	0.8141	0.7845	0.7518	0.7732
$t_6 = 0.1200$	0.8403	0.8216	0.7926	0.7605	0.7754
$t_7 = 0.1400$	0.8456	0.8289	0.8006	0.7690	0.7775
$t_8 = 0.1600$	0.8508	0.8361	0.8084	0.7774	0.7797
$t_9 = 0.1800$	0.8559	0.8430	0.8160	0.7856	0.7818
$t_{10} = 0.2000$	0.8608	0.8498	0.8234	0.7937	0.7840

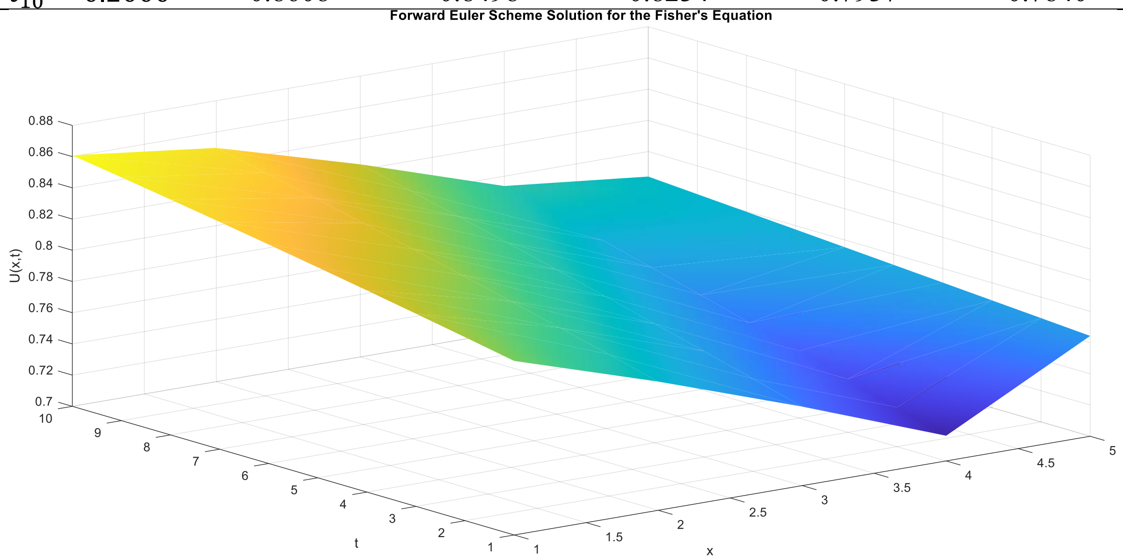

Fig.5: *Example 3a: The explicit scheme results when* $x \in [0,1], t \in [0,0.2], \beta = 1, h = 0.2, k = 0.02$ *and* $\delta = 3$.

From Table 5, the values are higher than in Example 2, confirming that increasing δ increases the solution values. The figure 5 shows a faster evolution of the solution over time.

Table 6: Example 3b: The Crank-Nicholson scheme results when $x \in [0,1], t \in [0,0.2], \beta = 1, h = 0.2, k = 0.02$ and $\delta = 3$.

	$x_1 = 0.200$	$x_2 = 0.400$	$x_3 = 0.600$	$x_4 = 0.800$	$x_5 = 1.000$
$t_1 = 0.0200$	0.8115	0.7825	0.7506	0.7162	0.7643
$t_2 = 0.0400$	0.8176	0.7906	0.7593	0.7252	0.7665
$t_3 = 0.0600$	0.8235	0.7986	0.7678	0.7342	0.7688
$t_4 = 0.0800$	0.8292	0.8065	0.7762	0.7431	0.7710
$t_5 = 0.1000$	0.8348	0.8141	0.7845	0.7518	0.7732
$t_6 = 0.1200$	0.8403	0.8216	0.7926	0.7605	0.7754
$t_7 = 0.1400$	0.8456	0.8289	0.8006	0.7690	0.7775
$t_8 = 0.1600$	0.8508	0.8361	0.8084	0.7774	0.7797
$t_9 = 0.1800$	0.8559	0.8430	0.8160	0.7856	0.7818
$t_{10} = 0.2000$	0.8608	0.8498	0.8234	0.7937	0.7840

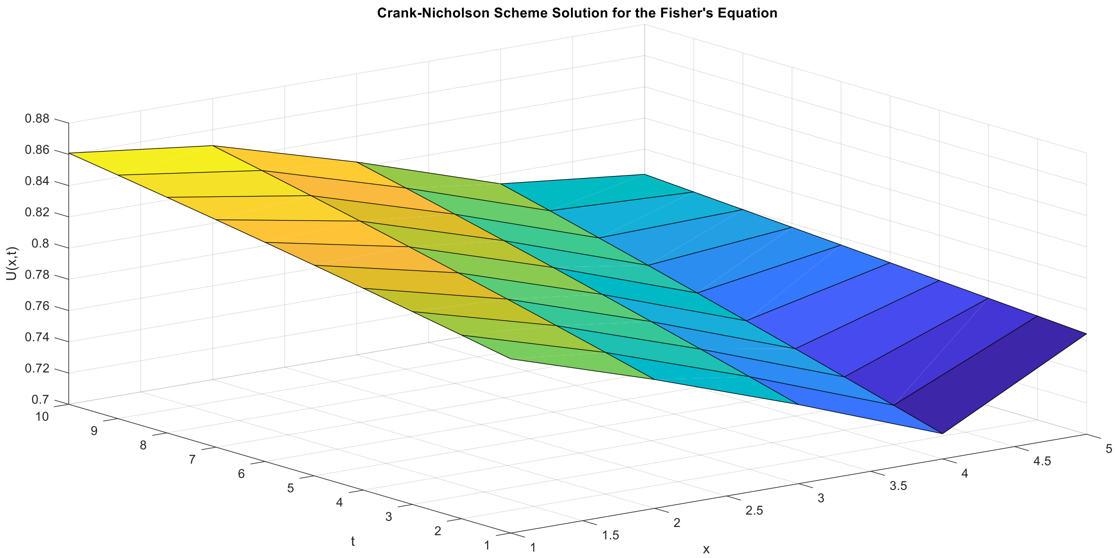

Fig.6: *Example 3b: The Crank-Nicholson scheme results when $x \in [0,1], t \in [0,0.2], \beta = 1, h = 0.2, k = 0.02$ and $\delta = 3$.*

The values in the table show a gradual increase with time for each spatial point (x_1 to x_5). At $x = 1.0$, the values are slightly higher than at other points, which may reflect a boundary effect. The curves trend upward steadily, indicating that the solution is stable. The figure shows that as x increases, the solution value decreases slightly (except for x = 1.0, where the values are higher) and the slope is larger compared to $\delta = 1$ or 2, confirming that increasing δ accelerates the system's dynamics.

Example 4

Take $x \in [0,1], t \in [0,0.2], \beta = 1, \delta = 4$ by divided $[0,1]$ at $n = 5$, the time interval $[0,0.2]$ is split into $m = 10$ sub-intervals with step size $k = 0.02$, and the step size on x-axis $h = 0.2$.

Table 7: Example 4a: The explicit scheme results when $x \in [0,1], t \in [0,0.2], \beta = 1, h = 0.2, k = 0.02$ and $\delta = 4$.

	$x_1 = 0.200$	$x_2 = 0.400$	$x_3 = 0.600$	$x_4 = 0.800$	$x_5 = 1.000$
$t_1 = 0.0200$	0.8550	0.8266	0.7946	0.7591	0.8129
$t_2 = 0.0400$	0.8611	0.8354	0.8041	0.7692	0.8150
$t_3 = 0.0600$	0.8669	0.8440	0.8135	0.7792	0.8171
$t_4 = 0.0800$	0.8725	0.8523	0.8226	0.7891	0.8192
$t_5 = 0.1000$	0.8780	0.8604	0.8316	0.7987	0.8212
$t_6 = 0.1200$	0.8833	0.8682	0.8402	0.8082	0.8233
$t_7 = 0.1400$	0.8884	0.8757	0.8487	0.8175	0.8253
$t_8 = 0.1600$	0.8933	0.8829	0.8568	0.8265	0.8273
$t_9 = 0.1800$	0.8980	0.8898	0.8647	0.8353	0.8293

$t_{10} = 0.2000$	0.9025	0.8964	0.8724	0.8439	0.8313

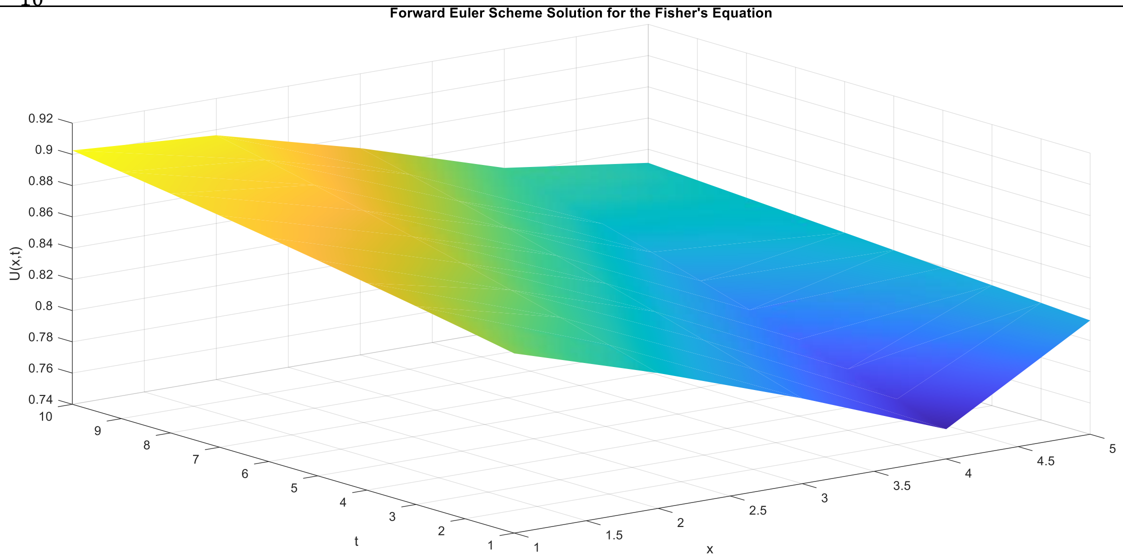

Fig.7: *Example 4a: The explicit scheme results when* $x \in [0,1], t \in [0,0.2], \beta = 1, h = 0.2, k = 0.02$ *and* $\delta = 4$.

From Table 7 the values continue to increase as δ increases. The figure shows a faster behavior of the solution, with higher values at x = 1.

Table 8: Example 4b: The Crank-Nicholson scheme results when $x \in [0,1], t \in [0,0.2], \beta = 1, h = 0.2, k = 0.02$ and $\delta = 4$.

	$x_1 = 0.200$	$x_2 = 0.400$	$x_3 = 0.600$	$x_4 = 0.800$	$x_5 = 1.000$
$t_1 = 0.0200$	0.8550	0.8266	0.7946	0.7591	0.8129
$t_2 = 0.0400$	0.8611	0.8354	0.8041	0.7692	0.8150
$t_3 = 0.0600$	0.8669	0.8440	0.8135	0.7792	0.8171
$t_4 = 0.0800$	0.8725	0.8523	0.8226	0.7891	0.8192
$t_5 = 0.1000$	0.8780	0.8604	0.8316	0.7987	0.8212
$t_6 = 0.1200$	0.8833	0.8682	0.8402	0.8082	0.8233
$t_7 = 0.1400$	0.8884	0.8757	0.8487	0.8175	0.8253
$t_8 = 0.1600$	0.8933	0.8829	0.8568	0.8265	0.8273
$t_9 = 0.1800$	0.8980	0.8898	0.8647	0.8353	0.8293
$t_{10} = 0.2000$	0.9025	0.8964	0.8724	0.8439	0.8313

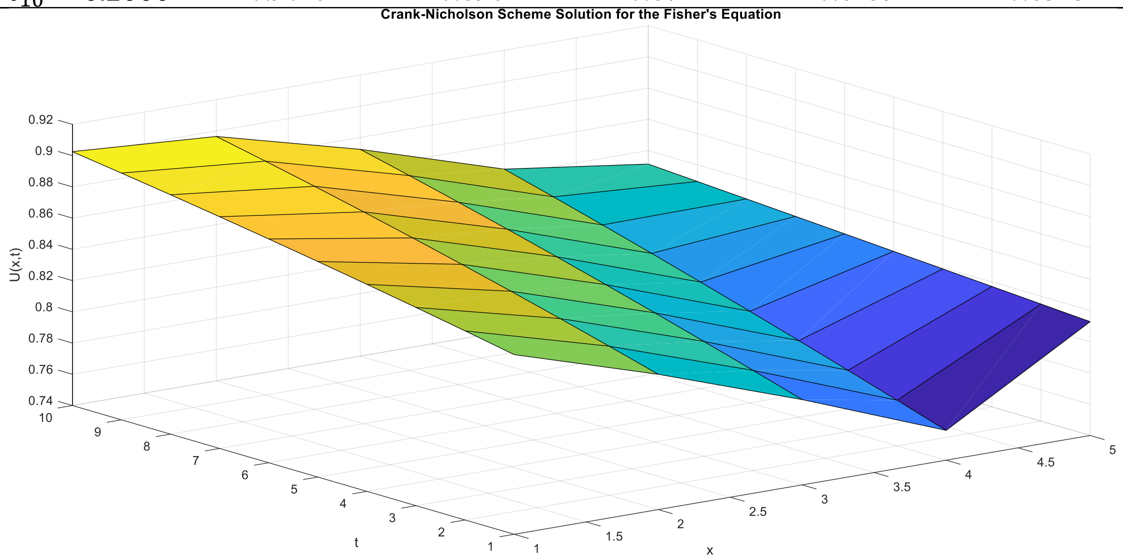

Fig.8: *Example 4b: The Crank-Nicholson scheme results when* $x \in [0,1], t \in [0,0.2], \beta = 1, h = 0.2, k = 0.02$ *and* $\delta = 4$.

The values in the table show the same solutions as those in Table 6, but the solution at x = 1.0 shows larger jumps, which may be due to the interaction of the terms with the parameter δ.

Example 5

Take $x \in [0,1], t \in [0,0.2], \beta = 1, \delta = 5$ by divided $[0,1]$ at $n = 5$, the time interval $[0,0.2]$ is split into $m = 10$ sub-intervals with step size $k = 0.02$, and the step size on x-axis $h = 0.2$.

Table 9: Example 5a: The explicit scheme results when $x \in [0,1], t \in [0,0.2]$, $\beta = 1, h = 0.2, k = 0.02$ and $\delta = 5$.

	$x_1 = 0.200$	$x_2 = 0.400$	$x_3 = 0.600$	$x_4 = 0.800$	$x_5 = 1.000$
$t_1 = 0.0200$	0.8822	0.8547	0.8229	0.7870	0.8445
$t_2 = 0.0400$	0.8882	0.8640	0.8332	0.7979	0.8465
$t_3 = 0.0600$	0.8939	0.8730	0.8431	0.8087	0.8485
$t_4 = 0.0800$	0.8994	0.8816	0.8528	0.8193	0.8504
$t_5 = 0.1000$	0.9047	0.8898	0.8622	0.8297	0.8524
$t_6 = 0.1200$	0.9098	0.8977	0.8712	0.8397	0.8543
$t_7 = 0.1400$	0.9146	0.9052	0.8799	0.8495	0.8562
$t_8 = 0.1600$	0.9192	0.9123	0.8882	0.8590	0.8581
$t_9 = 0.1800$	0.9236	0.9190	0.8961	0.8681	0.8599
$t_{10} = 0.2000$	0.9277	0.9253	0.9037	0.8769	0.8618

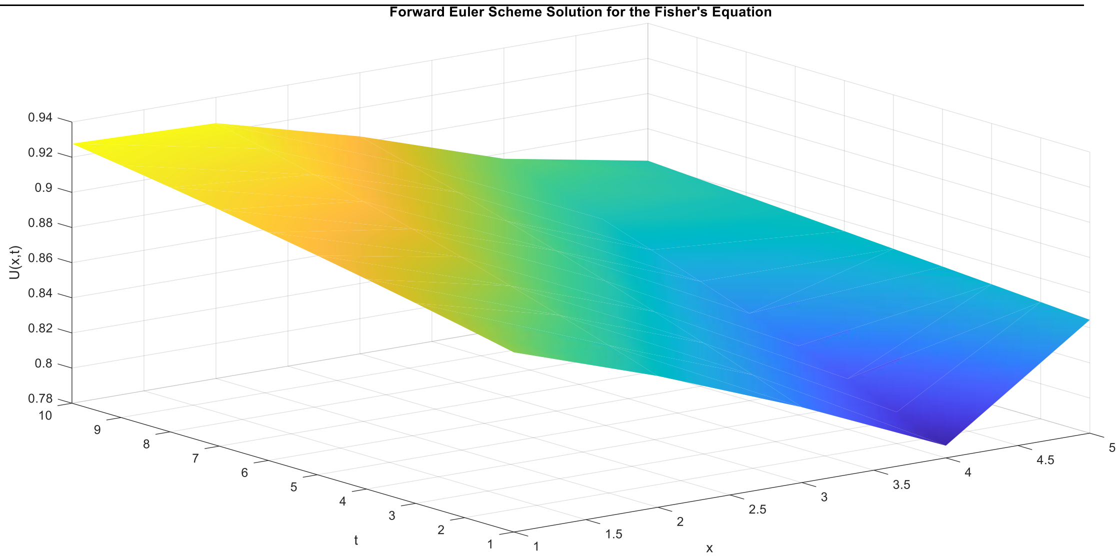

Fig.9: *Example 5a: The explicit scheme results when $x \in [0,1], t \in [0,0.2], \beta = 1, h = 0.2$, $k = 0.02$ and $\delta = 5$.*

From Table 9 the highest values so far are due to increasing δ. The figure shows a rapid evolution of the solution with time.

Table 10: Example 5b: The Crank-Nicholson scheme results when $x \in [0,1], t \in [0,0.2], \beta = 1, h = 0.2, k = 0.02$ and $\delta = 5$.

	$x_1 = 0.200$	$x_2 = 0.400$	$x_3 = 0.600$	$x_4 = 0.800$	$x_5 = 1.000$
$t_1 = 0.0200$	0.8822	0.8547	0.8229	0.7870	0.8445
$t_2 = 0.0400$	0.8882	0.8640	0.8332	0.7979	0.8465
$t_3 = 0.0600$	0.8939	0.8730	0.8431	0.8087	0.8485
$t_4 = 0.0800$	0.8994	0.8816	0.8528	0.8193	0.8504
$t_5 = 0.1000$	0.9047	0.8898	0.8622	0.8297	0.8524
$t_6 = 0.1200$	0.9098	0.8977	0.8712	0.8397	0.8543
$t_7 = 0.1400$	0.9146	0.9052	0.8799	0.8495	0.8562
$t_8 = 0.1600$	0.9192	0.9123	0.8882	0.8590	0.8581
$t_9 = 0.1800$	0.9236	0.9190	0.8961	0.8681	0.8599
$t_{10} = 0.2000$	0.9277	0.9253	0.9037	0.8769	0.8618

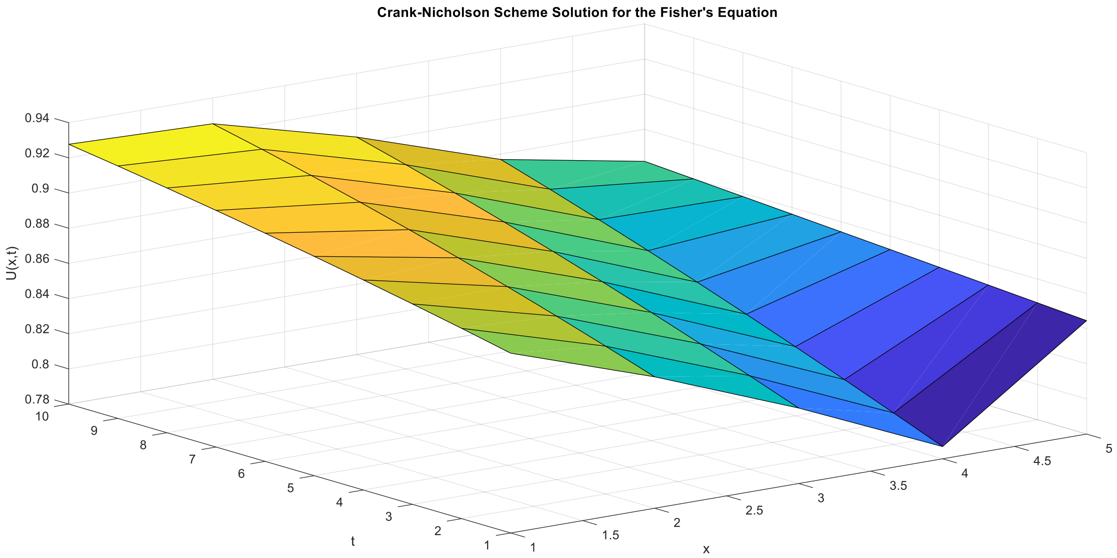

Fig.10: *Example 5b: The Crank-Nicholson scheme results when* $x \in [0,1]$, $t \in [0, 0.2]$, $\beta = 1$, $h = 0.2$, $k = 0.02$ *and* $\delta = 5$.

The values in Table 12 show that at $x = 1.0$, the values approach 0.9, which may indicate saturation or possible instability at δ.

We conclude from the above that the higher δ, the faster the solution values and the faster it evolves over time. Smaller steps ($h = 0.1$) give more accurate and smoother results. The performance of schemes, the explicit and Crank-Nicholson, give close results, indicating their stability for this equation. When $x = 1$, the values sometimes exhibit different behaviour, which reflect the influence of boundary conditions. Therefore, numerical solutions to the equation depend significantly on parameters δ and step size. Increasing δ increases solution values and speed of its evolution, while smaller steps lead to higher accuracy. Both schemes are effective for solving this equation under tested conditions.

Conclusions

GFE is a powerful tool for modelling the interactions between diffusion and non-linear growth, and its applications are broad in fields. The convergence conditions for explicit and Crank-Nicholson methods were analysed, and both methods demonstrated good stability in under certain conditions, with Crank-Nicholson method being superior in accuracy. Increasing value of leads to larger solution values and faster evolution, confirming its significant impact on the dynamics of the system. Small-step accuracy: Using small steps in time and space improves the accuracy of the results and reduces numerical errors. Both the explicit and Crank-Nicholson methods demonstrate effectiveness in solving the equation, with the explicit method being easier to implement, while the Crank-Nicholson method provides higher accuracy.

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الحل العددي لمعادلة فيشر المعممة باستخدام طريقتي أويلر الصريحة وكرانك-نيكلسون

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معلومات المؤلف
الايميل:

الخلاصة:
تقدم هذه الدراسة حلاً عددياً لمعادلة فيشر المعممة (GFE) باستخدام طريقتين للفروق المحدودة: الطريقة الصريحة وطريقة كرانك-نيكلسون. تم تحليل تقارب كل طريقة نظرياً وعملياً، مع دراسة تأثير المعامل δ على ديناميكيات النظام. أظهرت النتائج أن: زيادة δ تؤدي إلى تسارع في تطور الحل وارتفاع قيمه كما اظهرت الطريقة الصريحة أكثر كفاءة حسابياً، بينما تفوقت طريقة كرانك-نيكلسون في الدقة والاستقرار. استخدام خطوات زمنية ومكانية صغيرة يحسن دقة النتائج بشكل ملحوظ، تم التحقق من صحة النتائج عبر أمثلة عددية متنوعة باستخدام MATLAB R2022a، مما يؤكد فعالية الطريقتين في نمذجة الظواهر المعقدة التي تصفها المعادلة.