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Operational Spline Scaling Functions Method for Solving Optimal Control Problems

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Abstract

In this paper, a general expression formula for Spline Scaling Functions (SSFs) operational matrix of derivative is constructed. Then it is used to study a new iterative parameterization direct technique for treating optimal control problems approximately. Optimal control problems describe several important phenomena in mathematical science. In the present technique, an operational matrix of derivatives for such functions is constructed for resolving the optimal control problem. Since the obtained operational matrix of derivative comprises, many zeros elements, it can bring about a numerical accurate result with high reliability of achieving the desired results. An attractive product operation matrix of SSFs with other properties is also included in this work. With solving some examples, the comparison with the actual solutions shows that our algorithm is acceptable.

Introduction

Optimal control problems have attracted numerous researchers in science and engineering [1-3]. They are essentially related to the identification of state trajectories for a dynamical system over a time interval that optimizes a specific performance index, by achieving the best possible outcome through endogenous control of a parameter within a mathematical model of the system itself. The associated problem is characterized by a cost or objective function, depending on both the state and control variables, as well as by a group of constraints. Optimal control problems are typically nonlinear and hence do not admit analytic solutions. Therefore; several authors have suggested various techniques providing a numerical solution. different basis polynomials are considered for solving optimal control problems, such as, shifted Chebyshev polynomials [4-6], Chebyshev wavelets [7, 8], Legendre orthonormal basis [9], Taylor wavelets [10], orthonormal Bernstein polynomials [11], spline basis functions [12-16] and modified Hermite polynomials [17]. In addition, the scaling and wavelets functions play an important role in areas of mathematics. These functions have been utilized in the solution of differential equations, integral equations and approximation theory [18-20].

In this paper, novel approach based on Spline Scaling Functions (SSFs) operational matrices with their properties is applied for approximate solution of the optimal control problem. The advantage of operational matrices is to convert the original problem to a system of algebraic equations and then the integration product and differentiation will be eliminated with the aide of operational matrix of integration operational matrix of product and operational matrix of derivative respectively. In a result the complexity reduction can be obtained.

The main important goal in this work is to study the interesting properties SSFs and derived some new basic formulations of them. Two important operational matrices are devoted and their exact expression formulas are determined. In addition some useful lemmas concerning Basic Spline Scaling Functions are also presented. The polynomials and wavelets expansions together with operational matrices can be employed to solve problems in applied science and other fields of approximation theory.

Structure of the article

In section 3, Spline Scaling Functions are introduced and some new important properties concerning SSFs are derived. We obtain two important operational matrices for Basic Spline Scaling Functions: named operational matrix of derivative and product operational matrix for SSFs. They are drowned through section 3. Section 4, introduces an algorithm for solving optimal control problems based on SSFs. Some test examples are included in section 5 to confirm the efficiency of the proposed method. In section 6, some conclusions are listed.

The Definition of Spline Scaling Functions

The **Spline Scaling Functions** $\eta_{i,n}^{m}(t)$ can be defined on the interval $[0,1)$ as below

$$\eta_{i,n}^{m}(t) = \begin{cases} BS_i^m(kt-n) & t \in \left[\frac{n}{k}, \frac{n+1}{k}\right) \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where

$$BS_i^m(t) = \begin{cases} tBS_{i-1}^{m-1}(t) + (1-t)BS_i^{m-1}(t) & m > i \\ 0 & \text{for } i<0 \text{ or } i>m \end{cases} \tag{2}$$

For $i = 0, 1, \dots, m$ and the four arguments n, k, m, t are

(1) The translation argument $n = 0, 1, \dots, k-1$
(2) The number of partitions on $[0, 1]$, $k > 1$ is any positive integer
(3) The normalized time t
(4) The order of Bernstein polynomial on $[0, 1]$ is m

The polynomials in Eq. (2) are in Hilbert space $L^2[0, 1]$. It means that they form a complete set and are non-orthogonal.

The Spline Scaling Functions have many interesting properties; some of them are useful in our work

- $BS_i^m(t)BS_j^m(t) = \dfrac{\binom{m}{i}\binom{m}{j}}{\binom{2m}{i+j}} BS_{i+j}^{2m}(t) \qquad i, j \neq 0$
- $BS_i^m(t) = \sum_{j=0}^{k} \dfrac{\binom{m}{i}\binom{k}{j}}{\binom{m+k}{i+j}} BS_{i+j}^{m+k}(t) \qquad k, j = 0,1, \dots, m$

- $\frac{BS_i^m(t)}{\binom{m}{i}} = \frac{BS_i^{m+1}(t)}{\binom{m+1}{i}} + \frac{BS_{i+1}^{m+1}(t)}{\binom{m+1}{i+1}} \qquad i = 0,1, \dots, m$

Function Approximation Using SSFs

A function $x(t) \epsilon L^2[0,1]$ can be expanded in terms of SSFs as

$$x^m(t) = \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} x_{i,n} \; \eta_{i,n}^m(t) \tag{3}$$

If we consider truncated series in Eq. (3), one can get

$$x^m(t) = \sum_{n=0}^{k-1} \sum_{i=0}^{m} x_{i,n} \; \eta_{i,n}^m(t) \tag{4}$$

where

$$\eta^m(t) = \left[\eta_{00}, \eta_{10}, \dots, \eta_{m0}, \eta_{01}, \eta_{11}, \dots, \eta_{m1}, \dots, \eta_{0,k-1} \eta_{1,k-1}, \dots, \eta_{m,k-1}\right]^T \tag{5}$$

and x is a $(km \times 1)$ vector of unknown coefficients.

Properties of SSFs

Operational Matrix of Derivative

This section gives a new method for constructing operational matrix of derivatives for Spline Functions. We choose $m = 4$ and $k = 2$ and perform Eq. (1) to calculate the ten SSFs as illustrated in Table 1.

Table 1: The SSFs for $m = 4$ and $k = 2$

$0 \le t < \frac{1}{2}$	$\frac{1}{2} \le t < 1$
$\eta_{00}^4(t) = (1-2t)^4$	$\eta_{01}^4(t) = (2-2t)^4$
$\eta_{10}^4(t) = 8t(1-2t)^3$	$\eta_{11}^4(t) = 4t(2t-1)(2-2t)^2$
$\eta_{20}^4(t) = 24t^2(1-2t)^2$	$\eta_{21}^4(t) = 6(2t-1)^2(2-2t)^2$
$\eta_{30}^4(t) = 32t^3(1-2t)$	$\eta_{31}^4(t) = 4(2-2t)(2t-1)^3$
$\eta_{40}^4(t) = 16t^4$	$\eta_{41}^4(t) = (2t-1)^4$

By differentiating the ten basis in Table 1, one can get the new equations as shown in Table 2.

Table 2: The Derivatives SSFs for $m = 4$ and $k = 2$

$0 \le t < \frac{1}{2}$	$\frac{1}{2} \le t < 1$
$\dot{\eta}_{00}^4(t) = -8\eta_{00}^3(t)$	$\dot{\eta}_{01}^4(t) = -8\eta_{01}^3(t)$
$\dot{\eta}_{10}^4(t) = 8\left(\eta_{00}^3(t) - \eta_{10}^3(t)\right)$	$\dot{\eta}_{11}^4(t) = 8\left(\eta_{01}^3(t) - \eta_{11}^3(t)\right)$
$\dot{\eta}_{20}^4(t) = 8\left(\eta_{10}^3(t) - \eta_{20}^3(t)\right)$	$\dot{\eta}_{21}^4(t) = 8\left(\eta_{11}^3(t) - \eta_{21}^3(t)\right)$
$\dot{\eta}_{30}^4(t) = 8\left(\eta_{20}^3(t) - \eta_{30}^3(t)\right)$	$\dot{\eta}_{31}^4(t) = 8\left(\eta_{21}^3(t) - \eta_{31}^3(t)\right)$
$\dot{\eta}_{40}^4(t) = 8\eta_{30}^3(t)$	$\dot{\eta}_{41}^4(t) = 8\eta_{31}^3(t)$

The equations in Table 2 can be written in the following form

$$\dot{\eta}^4(t) = D\eta^3(t) \tag{6}$$

where

$\dot{\eta}^4(t) = [\dot{\eta}_{00}^4(t)\, \dot{\eta}_{10}^4(t)\;\; \dot{\eta}_{20}^4(t)\, \dot{\eta}_{30}^4(t)\;\; \dot{\eta}_{40}^4(t)\;\; \dot{\eta}_{01}^4(t)\;\; \dot{\eta}_{11}^4(t)\, \dot{\eta}_{21}^4(t)\;\; \dot{\eta}_{31}^4(t)\, \dot{\eta}_{41}^4(t)\,]^T$

$\eta^3(t) = [\eta_{00}^3(t)\, \eta_{10}^3(t)\;\; \eta_{20}^3(t)\;\; \eta_{30}^3(t)\;\; \eta_{01}^3(t)\;\; \eta_{11}^3(t)\, \eta_{21}^3(t)\, \eta_{31}^3(t)\,]^T$

The operational matrix D can be defined as

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_1 \end{pmatrix} \text{ where } D_1 = 2^3 \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The general formula for operational matrix of derivatives for SSFs are built through the following theorem.

Theorem 1. Let $\eta^m(t)$ be the SSFs vector defined in Eq. 5. The derivative of the vector $\eta^m(t)$ can be given by

$$\dot{\eta}_{in}^m(t) = \begin{cases} k^m(\,\eta_{i-1,n}^{m-1} - \eta_{in}^{m-1}) & i = 0,1,\dots,m-1 \\ k^m\,\eta_{m-1,n}^{m-1} & i = m \end{cases} \tag{7}$$

where $n = 0, 1, \dots, k-1$

Proof. The nth element of the SSFs vector $\eta^m(t)$ in Eq. 5 can be written as

$$\varphi_r^m(t) = \eta_{in}^m(t) = B_i^m(kt-n)\chi_{\left[\frac{n}{k},\frac{n+1}{k}\right]},\; r = 1,2,\dots,m$$

where

$$\chi_{\left[\frac{n}{k},\frac{n+1}{k}\right]} = \begin{cases} 1 & \frac{n}{k} \le t \le \frac{n+1}{k} \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

and $r = n(m+1) + (i+1)$

Take the derivative of Eq. 8 with respect to t yields

$$\frac{d\varphi_r}{dt} = \frac{d}{dt} BS_i^m(kt-n)\chi_{\left[\frac{n}{k},\frac{n+1}{k}\right]} \tag{9}$$

Using the known relation of the differentiation of SSFs,

$$\frac{d}{dt} BS_i^m(t) = m\left(BS_{i-1}^{m-1}(t) - B_{Si}^{m-1}(t)\right) \qquad \text{for} \quad 0 \le i \le m \tag{10}$$

Therefore;

$$\frac{d\varphi_r(t)}{dt} = k^m\left(BS_{i-1}^{m-1}(kt-n) - BS_i^{m-1}(kt-m)\right)\chi_{\left[\frac{n}{k},\frac{n+1}{k}\right]} \tag{11}$$

with the use of Eq. (1) we will get the required results.

Product Operational Matrix for SSFs

An important operator which plays a very important role in modeling some equations can be defined by

$$\eta^m(t)\eta^m(t)^T = P\eta(t) \qquad (12)$$

where the matrix P is a $2(m+1)\times 2(m+1)$ named a product operational matrix of SSFs. To illustrate the formulation of the product matrix, $m=2$, $k=2$ are selected in Eq. 1

The entries of the matrix $\eta^2(t)\eta^2(t)^T$ can be written as follows

$$\eta^2(t)\eta^2(t)^T = \begin{pmatrix} \eta_{00}^2\eta_{00}^2 & \eta_{00}^2\eta_{01}^2 & \eta_{00}^2\eta_{02}^2 & \eta_{00}^2\eta_{10}^2 & \eta_{00}^2\eta_{11}^2 & \eta_{00}^2\eta_{12}^2 \\ \eta_{01}^2\eta_{00}^2 & \eta_{01}^2\eta_{01}^2 & \eta_{01}^2\eta_{02}^2 & \eta_{01}^2\eta_{10}^2 & \eta_{01}^2\eta_{11}^2 & \eta_{01}^2\eta_{12}^2 \\ \eta_{02}^2\eta_{00}^2 & \eta_{02}^2\eta_{01}^2 & \eta_{02}^2\eta_{02}^2 & \eta_{02}^2\eta_{10}^2 & \eta_{02}^2\eta_{11}^2 & \eta_{02}^2\eta_{12}^2 \\ \eta_{10}^2\eta_{00}^2 & \eta_{10}^2\eta_{01}^2 & \eta_{10}^2\eta_{02}^2 & \eta_{10}^2\eta_{10}^2 & \eta_{10}^2\eta_{11}^2 & \eta_{10}^2\eta_{12}^2 \\ \eta_{11}^2\eta_{00}^2 & \eta_{11}^2\eta_{01}^2 & \eta_{11}^2\eta_{02}^2 & \eta_{11}^2\eta_{10}^2 & \eta_{11}^2\eta_{11}^2 & \eta_{11}^2\eta_{12}^2 \\ \eta_{12}^2\eta_{00}^2 & \eta_{12}^2\eta_{01}^2 & \eta_{12}^2\eta_{02}^2 & \eta_{12}^2\eta_{10}^2 & \eta_{12}^2\eta_{11}^2 & \eta_{12}^2\eta_{12}^2 \end{pmatrix}$$

where

$\eta_{00}^2\eta_{00}^2 = \eta_{00}^4$ $\qquad$ $\eta_{10}^2\eta_{10}^2 = \eta_{01}^4$

$\eta_{00}^2\eta_{01}^2 = \frac{1}{2}\eta_{01}^4$ $\qquad$ $\eta_{10}^2\eta_{11}^2 = \frac{1}{2}\eta_{11}^4$

$\eta_{00}^2\eta_{02}^2 = \frac{1}{6}\eta_{02}^4$ $\qquad$ $\eta_{10}^2\eta_{12}^2 = \frac{1}{6}\eta_{21}^4$

$\eta_{01}^2\eta_{01}^2 = \frac{1}{2}\eta_{02}^4$ $\qquad$ $\eta_{11}^2\eta_{11}^2 = \frac{1}{2}\eta_{21}^4$

$\eta_{01}^2\eta_{02}^2 = \frac{1}{2}\eta_{03}^4$ $\qquad$ $\eta_{11}^2\eta_{12}^2 = \frac{1}{2}\eta_{31}^4$

$\eta_{02}^2\eta_{02}^2 = \eta_{04}^4$ $\qquad$ $\eta_{12}^2\eta_{12}^2 = \eta_{41}^4$

Note that $\eta_{in}^m\eta_{jk}^m = 0$ for $n \neq j$ because the entries of $\eta^m(t)$ are the interval $\frac{n}{2^k} \leq t \leq \frac{n+1}{2^{k+1}}$

$$\eta^2(t)\eta^2(t)^T = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}$$

where

$$R_1 = \begin{pmatrix} \eta_{00}^4 & \frac{1}{2}\eta_{10}^4 & \frac{1}{6}\eta_{20}^4 \\ \frac{1}{2}\eta_{10}^4 & \frac{1}{2}\eta_{20}^4 & \frac{1}{2}\eta_{30}^4 \\ \frac{1}{6}\eta_{20}^4 & \frac{1}{2}\eta_{30}^4 & \eta_{40}^4 \end{pmatrix}, \text{ and } R_2 = \begin{pmatrix} \eta_{01}^4 & \frac{1}{2}\eta_{11}^4 & \frac{1}{6}\eta_{21}^4 \\ \frac{1}{2}\eta_{11}^4 & \frac{1}{2}\eta_{21}^4 & \frac{1}{2}\eta_{31}^4 \\ \frac{1}{6}\eta_{21}^4 & \frac{1}{2}\eta_{31}^4 & \eta_{41}^4 \end{pmatrix}$$

or $\eta^2(t)\eta^2(t)^T = p_{6\times 6}\,\eta^4(t)$

where $p_{6\times6} = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$, $(P_i)_{3\times3} = \begin{pmatrix} P_{0i} & \frac{1}{2}P_{1i} & \frac{1}{6}P_{2i} \\ \frac{1}{2}P_{1i} & \frac{1}{2}P_{2i} & \frac{1}{2}P_{3i} \\ \frac{1}{6}P_{2i} & \frac{1}{2}P_{3i} & P_{4i} \end{pmatrix}$

That is for $i = 0,1$.

In general, the operational matrix of the product can be given by

$$\eta_{ni}^{m}(t)\,\eta_{nk}^{m}(t) = \frac{\binom{m}{i}\binom{m}{k}}{\binom{2m}{i+k}}\eta_{n,i+k}^{2m}(t)$$

$$\eta_{in}^{m}(t)\,\eta_{jk}^{m}(t) = 0 \quad \text{for} \quad i \neq j$$

Let $r = n(m+1) + i + 1$ then $\eta_{ni=}^{m}\eta_{r}^{m}$
Therefore;

$$\eta_{r_1}^{m}\eta_{r_2}^{m} = \frac{\binom{m}{(r_1-1)+n(m+1)}\binom{m}{(r_2-1)+n(m+1)}}{\binom{2m}{(r_1+r_2-2)+nm}}\eta_{(r_1+r_2)-(n+1)}^{2m}$$

Further properties of SSFs

Some exact relations between SSFs and their derivatives are listed throughout the following lemmas.

Lemma 1. The SSFs of order $m-1$ can be expressed by SSFs of order m by the following formula

$$\eta_{in}^{m-1}(t) = \frac{1}{m}\Big((m-i)\eta_{in}^{m}(t) + (i+1)\eta_{i+1,n}^{m}(t)\Big) \qquad (13)$$

for $i = 0,1,\dots,m$ and $n = 0,1,\dots,k-1$

Proof. Using the following formula

$$\frac{1}{\binom{m}{i}}\eta_{in}^{m}(t) + \frac{1}{\binom{m}{i+1}}\eta_{i+1,n}^{m}(t) = \frac{1}{\binom{m-1}{i}}\eta_{i,n-1}^{m}(t)$$

We can write arbitrary SSFs in terms of SSFs of higher degree,

That is, $\eta_{in}^{m-1}(t) = \binom{m-1}{i}\left(\frac{1}{\binom{m}{i}}\eta_{in}^{m}(t)\right) + \frac{1}{\binom{m}{i+1}}\eta_{i+1,n}^{m}(t)$

In other words

$$\eta_{in}^{m-1}(t) = \frac{1}{m}\Big((m-i)\eta_{in}^{m}(t) + (i+1)\eta_{i+1,n}^{m}(t)\Big)$$

As we want to prove.

Remark 1: Note that Eq. 13 can be written in the following compact form

$$\eta_{in}^{m-1}(t) = R\,\eta^{m}(t)$$

where $R = \begin{pmatrix} R_1 & 0 \\ 0 & R_1 \end{pmatrix}$ is a $2m \times 2(m+1)$ matrix and R_1 is $m \times (m+1)$ matrix. For example for $m = 4$, yields

$$R_1 = \frac{1}{2^2}\begin{pmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

Lemma 2. The first derivative of SSFs in terms of SSFs in the same order is formulated as

$$\dot{\eta}_{in}^{m}(t) = 2\left((m-i+1)\eta_{i-1,n}^{m}(t) + (2i-m)\eta_{i,n}^{m}(t) + (i+1)\eta_{i+1,n}^{m}(t)\right) \qquad (14)$$

for $i = 0,1,\dots,m$ and $n = 0,1,\dots,k-1$

Proof. Since the first derivative of SSFs is given by Eq. 7

$$\dot{\eta}_{in}^{m}(t) = 2m\left(\eta_{i-1,n}^{m-1}(t) - \eta_{i,n}^{m-1}(t)\right)$$

and the formula of SSFs of order $m-1$ can be expressed by SSFs of order m using the formula:

$$\dot{\eta}_{in}^{m-1}(t) = \frac{1}{m}\left((m-i)\eta_{i,n}^{m}(t) + (i+1)\eta_{i+1,n}^{m}(t)\right)$$

By combining these two relations, one can obtain the required results.

Remark 2. The compact form of Eq. 14 can be formulated by the following

$$\dot{\eta}_{in}^{m}(t) = S\,\eta_{in}^{m}$$

where the matrices S and S_1 are of $2((m+1)\times(m+1))$ and $(m+1)\times(m+1)$ dimensions respectively. They are defined as below

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & S_1 \end{pmatrix}$$

The entries of the matrix S_1 can be obtained from the following relations

$$S_1 = (s_1)_{ij} = \begin{cases} -2m + 4(i-1) & i = j \\ 2m - 2(j-1) & i - j = 1,\ i > j \\ -2m - 2(i-4) & j - i = 1,\ i < j \\ 0 & \text{otherwise} \end{cases}$$

Application of approximating functions and operational matrices of SSFs to optimal control problem.

Consider the following optimal control problem minimize the functional

$$J = \int_0^1 F\big(t, x(t), u(t)\big)dt \qquad (15)$$

subject to $u(t) = F\big(t, x(t), \dot{x}(t)\big)$ (16)

with the boundary conditions $x(0) = x_0$ and $x(1) = x_1$ (17)

To solve the OCP (15-17) by means of SSFs and their properties, two examples are discussed

Example 1. Find an optimal control $u(t)$ with the cost functional minimize

$$J = \int_0^1 (x(t)^2 + u(t)^2)dt \qquad (18)$$

subject to $u(t) = \dot{x}(t)$ (19)

and boundary conditions $x(0) = 0, x(1) = 0.5$ (20)

Let the initial approximation of $x(t)$ is

$x^1(t) = x^0 + (x^1 - x^0)$ $(0.5\, \eta_{10}^1(t) + 0.5\eta_{10}^1(t) + \eta_{11}^1(t))$

By Eq. 6, one can get

The first approximation: $x^1(t) = d_1\eta^1(t)$,

The second approximation: $x^2(t) = d_1\eta^1(t) + a_2\ (\eta^2(t) - \eta^1(t))$, a_2=0.113636,

or $x^2(t)$ can be written as $x^2(t) = d_1\eta^1(t) + d_2\eta^2(t)$ (21)

The third approximation: $x^3(t) = x^2(t) + a_3 + \eta^3(t) - \eta^2(t)$, a_3 =0.029167

or $x^3(t) = d_1\eta^1(t) + d_2\eta^2(t) + d_3\eta^3(t)$

The fourth approximation: $x^4(t) = x^3(t) + a_4 + (\eta^4(t) - \eta^3(t), a_4 = 0.012606$

or $x^4(t) = d_1\eta^1(t) + d_2\eta^2(t) + d_3\eta^3(t) + d_4\eta^4(t)$

The control variable can be evaluated using Eq. 19 as follows

Rearranging Eq. 21 to get the desired form

$x^2(t) = d_1(M_1\eta^2(t)) + d_2(M_2\eta^3(t))$ (22)

where $\eta_1 = \begin{pmatrix} s_1 & 0 \\ 0 & s_1 \end{pmatrix}$, $\eta_2 = \begin{pmatrix} s_2 & 0 \\ 0 & s_2 \end{pmatrix}$, $S_1 = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix}$, $S_2 = \begin{pmatrix} 1 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} & 1 \end{pmatrix}$

$\dot{x}^2(t) = d_1(M_1D^1\eta^1(t)) + d_2(M_2D^2\eta^2(t))$ (23)

and $u^2(t) = e_1\eta^1(t) + e_2\eta^2(t)$

Similarly, the third approximation for $u(t)$ can be found as below

$u^3(t) = e_1\eta^1(t) + e_2\eta^2(t) + e_3\eta^3(t)$

$u_4(t) = e_1\eta^1(t) + e_2\eta^2(t) + e_3\eta^3(t) + e_4\eta^4(t)$

where, $\eta^1(t) = [\eta_{00}^1(t)\ \ \eta_{10}^1(t)\ \ \eta_{01}^1(t)\ \ \eta_{11}^1(t)]$,

and $\eta^2(t)$=$[\eta_{00}^2(t)\ \ \eta_{10}^2(t)\ \ \eta_{20}^2(t)\ \ \eta_{01}^2(t)\ \ \eta_{11}^2(t)\ \ \eta_{21}^2(t)\ ]$. Note that the values for the vectors d_i and e_i, $i = 1,2,3,4$ are listed in Table 3.

Table 3. The values of the vector d_i and e_i, $i = 1,2,3,4$, for Example 1

The values of the vector d_i				The values of the vector e_i			
d_1	d_2	d_3	d_4	e_1	e_2	e_3	e_4
0	0	0	0	0	-0.11364	0	0
0	-0.02841	0	0	0	-0.0568	-0.00972	0
0.25	-0.02841	0.00243	0	0.5	0.0560	-0.01215	-0.001576
0.5	-0.02841	0.00395	-0.000394	1	0.03684	-0.00729	-0.003152
	-0.02841	-0.00365	-0.00210			-0.00729	0.003152
	0	-0.00480	-0.000079			0.002431	-0.003152
			-0.001182			0.00972	-0.00315
			-0.001576			0.02917	-0.001576
			0				0.003152
			0				0.01261

For optimal values of the performance index J corresponding to $n = 1,2,3$, that is when $x(t) = x^1(t)$, $u(t) = u^1(t)$, $x(t) = x^2(t)$, $u(t) = u^2(t)$, $x(t) = x^3(t)$, $u(t) = u^3(t)$, Respectively, one can refer to Table 3.

Table 3. Approximate values of J for Example 1

Iteration	Our method	Error
1	0.3285984848	3.379×10^{-4}
2	0.3284769571	2.181×10^{-4}
3	0.3284627061	2.031×10^{-4}

Note the exact value of J is 0.3282588214.

Example 2. Consider the non-linear control system which consists of minimizing

$$J = \int_0^1 u^2(t)dt$$

Subject to $u(t) = \dot{x}(t) - x^2(t)sint$, $x(0) = 0$, $x(1) = 0.5$

In this example the initial approximation is $x^1(t) = d_1\eta^1(t)$

The second approximation is $x^2(t) = d_1\eta^1(t) + d_2\eta^2(t)$

$$u^2(t) = e_1\eta^1(t)+e_2\eta^2(t) - \left(d_1\eta^1(t) + d_2\eta^2(t)\right)^2 sin(t)$$

The third approximation is $x^3(t) = d_1\eta^1(t) + d_2\eta^2(t) + d_3\eta^3(t)$

$$u^3(t) = e_1\eta^1(t) + e_2\eta^2(t) + e_3\eta^3(t) - \left(d_1\eta^1(t) + d_2\eta^2(t) + d_3\eta^3(t)\right)^2 sin(t)$$

The fourth approximation is $x^4(t) = d_1\eta^1(t) + d_2\eta^2(t) + d_3\eta^3(t) + d_4\eta^4(t)$

$$u^4(t) = e_1\eta^1(t) + e_2\eta^2(t) + e_3\eta^3(t) + e_4\eta^4(t) - \left(d_1\eta^1(t) + d_2\eta^2(t) + d_3\eta^3(t) + d_4\eta^4(t)\right)^2 sin(t)$$

Note that the values for the vectors d_i and e_i, $i = 1,2,3,4$ are listed in Table 4.

Table 4. The values of the vector d_i and e_i, $i = 1,2,3,4$, for Example 2

The values of the vector d_i				The values of the vector e_i			
d_1	d_2	d_3	d_4	e_1	e_2	e_3	e_4
0	0	0	0	0	-0.0378	0	0
0	-0.0095	0	0	0	-0.0189	-0.0054	0
0.25	-0.0095	-0.00136	0	0.25	0	-0.00679	0.2625
0.5	-0.0095	-0.00204	0.0656	0.5	0	-0.00408	0.5240
	-0.0095	-0.00204	0.1312		0.0189	-0.00408	0.5240
	0	-0.00272	0.1312		0.0378	-0.00136	0.2625
		0.00273	0.19684			0.05433	0.2625
			0.2625			0.0163	0.2625
			0.2625				-0.5240
			0				-0.2097

The approximate values for J are listed in Table 5 together with the exact one.

Table 5. Approximate values of J for Example 2

Iteration	Our method	Absolute Error
1	0.2005	0.0000
2	0.2005	0.0000
3	0.2005	0.0000
Exact	0.2005	

Example 3. Consider third problem of minimizing

$$J = 0.5 \int_0^1 (3x^2(t) + u^2(t))dt$$

Subject to $u(t) = \dot{x}(t) + x(t)$, $x(0) = 0$, $x(1) = 2$. The exact value for the cost is $J = 6.1586$.

For optimal values of the performance index J corresponding to $n = 1, 2,3,$ one can refer to Table 6. The obtained results in [21] are also included in Table 6.

Table 6. Approximate values of J for Example 3.

Iteration	Our method	Method in [21]
1	6.1904	6.195
2	6.1575	6.1775
3	6.1548	6.1753

Conclusion

A recursive definition of the SSFs is presented, that is the SSFs of degree m can be expressed by two SSFs of degree $m-1$. A rasing degree formula is derived concerning SSFs. That is SSFs of degree $< m$ can be written as a linear combination of SSFs of degree m. Derivative of SSFs of degree m are functions of degree $m-1$. In this article an exact expression of SSFs of first derivative is formulated in terms of SSFs in the same order m. New exact matrices concerning

SSFs are derived named operational matrix of derivative, which differentiate the vectors of SSFs without any error; also a product operational matrix of SSFs is presented. By newly matrices many problems can be solved. They are applied together with a special direct technique to solve problems in optimal control. The efficiency of the proposed technique is shown from the obtained results. The aim of such technique is to get an effective algorithm that is suitable the digital computers by reducing the underlying optimal control problem to an optimization problem. This leads to a wide future work in solving many problems in science and engineering. Furthermore; they led to fewer errors when they were applied in solving the optimal control problem. These operational matrices can be utilized for approximate solution of calculus of variation problems, integral equations and other applications.

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طريقة عمليات دوال السبلاين المحجمة لحل مسائل السيطرة المثلى

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دوال السبلاين المحجمة(SSFs) ، مصفوفة العمليات، مسائل السيطرة المثلى، المسائل الخطية وغير الخطية، النتائج العددية.

معلومات المؤلف

الايميل:

الخلاصة:

في هذا البحث، تم بناء صيغة تعبير عامة عن مصفوفة عمليات المشتقات لدوال السبلاين المحجمة. ثم استخدمت لدراسة تقنية مباشرة جديدة لتحديد المعاملات التكرارية لمعالجة مسائل السيطرة المثلى بصورة تقريبية. تصف مسائل السيطرة المثلى عدة ظواهر مهمة في العلوم الرياضية. في التقنية الحالية، يتم إنشاء مصفوفة عمليات للمشتقات لمثل هذه الدوال لحل مسائل السيطرة المثلى. نظرًا لأن مصفوفة العمليات للمشتقات التي تم الحصول عليها تشتمل على العديد من عناصر الأصفار، فيمكنها تحقيق نتيجة رقمية دقيقة مع موثوقية عالية لتحقيق النتائج المرجوة. من خلال حل بعض الامثلة، تظهر المقارنة مع الحلول الفعلية أن الخوارزمية لدينا مقبولة.