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On a Subclass of Meromorphic multivalent Functions associated with q-differential operator

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Abstract

In this present paper, we have introduced and defined a new subclass of meromorphic multivalent function $Lk_{\mu,c}(v,m,A)$using the q-differential operator in punctured unit disk $U^* = \{z \in \mathbb{C}: 0 < |z| < 1\}$. We have obtained some geometric properties like coefficient inequality, a sufficient and necessary condition for the meromorphic multivalent functions to be in the class $Lk_{\mu,c}(v,m,A)$, and we have get a corollary from that theorem. Also, we have acquired the growth and distortion theorems bounds for the class $Lk_{\mu,c}(v,m,A)$. Subsequently we have fixed the radii of starlikeness and convexity theorems. After that we have proved the class $Lk_{\mu,c}(v,m,A)$, is closed under convex linear combinations. Also, we have introduced the extreme points theorem and proved that. Finally, we have obtained arithmetic mean theorem in the same class.

Mathematics Subject Classification: Primary 30C45

1. Introduction:

Let A_v indicate the class of meromorphic multivalent functions $h(z)$ normalized by the following:

$$h(z) = z^{-v} + \sum_{s=1}^{\infty} a_{s+v} + z^{s+v}, v \in N\backslash\{0\}. \qquad (1)$$

which are analytical in the unit disk with a hole in it

$$U^* = \{z \in \mathbb{C}: 0 < |z| < 1\} = N\backslash\{0\}.$$

In addition, a function h is said to belong to the class $LS_{\mathrm{P}}^*(\alpha)(0 \le \alpha < 1)$ of star-like meromorphic p-valent function

$$h(z) \in LS_{\mathrm{P}}^*(\alpha) \leftrightarrow \Re\frac{zh'(z)}{vh(z)} < -\alpha$$

A function h is therefore said to be a member of the class $LC_v^{(\alpha)}(0 \leq \alpha < 1)$ of star-like meromorphic p-valent function, if

$$h(z) \in LC_v(\alpha) \leftrightarrow \Re \frac{(zh'(z))'}{vh'(z)} < -\alpha$$

It is evident that the class of meromorphic p-valent starlike functions, $LS_{\mathrm{P}}^*(0) = LS_{\mathrm{P}}^*$, exists. The class of meromorphic p-valent close-to-convex functions is designated by $Lk_v^*(\alpha)$, and it is defined as:

$$h(z) \in Lk_{\mathrm{P}}^*(\alpha) \leftrightarrow \Re \frac{zh'(z)}{vg(z)} < -\alpha\ .$$

Where $g(z) \in LS_{\mathrm{P}}^*$ A function h, where $0 < c < 1$, has the following definition for the q-derivative (or q-difference) operator:

$$\delta_c h(z) = \frac{h(cz) - h(z)}{z(c-1)}\ \ , (z \neq 0).$$

It is clear to observe that

$$\delta_c\{\sum_{s=1}^{\infty} a_s z^s\} = \sum_{s=1}^{\infty} [s,c] a_s z^{s-1}\ , (s \in N\ , z \in D) \qquad (2)$$

The definition of the q-number shift factorial for every non-negative integer s is

$$[s,c]! = \begin{cases} 1, s = 0 \\ [1,c][2,c][3,c] \dots . [s,c],\ \ s \in N. \end{cases}$$

In addition, the q-generalized Pochhammer symbol for $x \in R$ is provided by

$$[x,c]_s = \begin{cases} 1, s = 0 \\ [x,c][x+1,c][x+2,c] \dots . [x+s-1,c], \qquad s \in N. \end{cases}$$

Comparative operator $D_{\mu,c}: A_v \rightarrow A_v$ is described as in [1]:

$$D_{\mu,c} h(z) = (1 + [v,c]\mu) h(z) + \mu c^v z \delta_c h(z). \qquad (3)$$

Where $\mu \geq 0$. One may quickly determine that using Equation (1)

$$D_{\mu,c} h(z) = z^{-v} + \sum_{s=1}^{\infty} (1 + [v,c]\mu + \mu c^v [s,c]) a_{s+v} z^{s+v}.$$

Where

$$D_{\mu,c}^0 h(z) = h(z).$$

And

$$D_{\mu,c}^2 h(z) = D_{\mu,c}\left(D_{\mu,c} h(z)\right)$$

$$= z^{-v} + \sum_{s=1}^{\infty} (1 + [v,c]\mu + \mu c^v [s,c])^2 a_{s+v} z^{s+v}. \qquad (4)$$

In a similar way for $m \in N$, we have

$$D^{m}{}_{\mu,c}h(z) = z^{-v} + \sum_{s=1}^{\infty}(1 + [v, c]\mu + \mu c^{v}[s, c])^{m} a_{s+v} z^{s+v}.$$

Which studied by [2] We create a subclass $Lk_{\mu,c}(v, m, A)$of A_v by utilizing the operator $D^{m}{}_{\mu,c}$ as follows. In this article, we are primarily motivated by the newly published study of Hu et al. in Symmetry **[3]** and some other relevant research as described above for example **[4]**, **[5].**

Now, let $-1 \leq B < A \leq 1$, $\forall z \in U^{*}$, a function $h \in A_v$ is referred to as a class member. $Lk_{\mu,c}(v, m, A)$ when it fulfills:

$$\left| \frac{z\left(\delta_c D_{\mu,c}^{m} h(z)\right)' + v\delta_c D_{\mu,c}^{m} h(z)}{Bz\left(\delta_c D_{\mu,c}^{m} h(z)\right)' + Av\delta_c D_{\mu,c}^{m} h(z)} \right| < 1, \qquad (5)$$

See for example **[6]**, **[7],** and **[8]**. Several writers examined the following intriguing geometric features of this function subclass for various classes, such as **[9]**, **[10]**, **[11]**, and **[12]**.

2. Sufficient and necessary condition:

Here, we provide a necessary and sufficient condition for the function h to belong to the class $Lk_{\mu,c}(v, m, A)$.

Theorem (2.1): $h(z)$ defined by (1) is in the class $Lk_{\mu,c}(v, m, A)$. if and only if

$$\sum_{s=0}^{\infty}(1 + [v, c]\mu + \mu c^{v}[s, v])^{m}[(s + v)(1 - B) + P(1 - A)]a_{s+v}$$

$$\leq v(A - B), \qquad (6)$$

where $-1 \leq B < A \leq 1$ and for all $z \in U^{*}$ .For the function, the outcome is sharp

$$h(z) = z^{-v} + \frac{v(A-B)}{(1+[v,c]\mu+\mu c^{v}[s,c])^{m}[(s+v)(1-B)+P(1-A)]} z^{s+v}, s \in \mathbb{N}$$

Proof: Let (6) holds true and $|z|$ equal one, then

$$\left| \frac{\sum_{s=0}^{\infty}(1+[v,c]\mu+\mu c^{v}[s,c])^{m}(s+2v)a_{s+v}z^{s+v}}{\frac{v(A-B)}{z^{v}} + \sum_{s=0}^{\infty}(1+[v,c]\mu+\mu c^{v}[s,c])^{m}((s+v)B+Av)a_{s+v}z^{s+v}} \right| < 1.$$

$$\frac{\left|\sum_{s=0}^{\infty}(1+[v,c]\mu+\mu c^{v}[s,c])^{m}(s+2v)a_{s+v}z^{s+v}\right|}{\left|\frac{v(A-B)}{z^{v}} + \sum_{s=0}^{\infty}(1+[v,c]\mu+\mu c^{v}[s,c])^{m}((s+v)B+Av)a_{s+v}z^{s+v}\right|} < 1.$$

$$\sum_{s=0}^{\infty}(1 + [v, c]\mu + \mu c^{v}[s, c])^{m}[(s + v)(1 - B) + P(1 - A)]|a_{s+v}| < v(A - B)$$

So, by principle of maximum modulus, $h \in Lk_{\mu,c}(v, m, A)$.

Conversely, suppose that $h \in Lk_{\mu,c}(v, m, A)$ Then from (5), we have

$$\left|\frac{\sum_{s=0}^{\infty}(1+[v,c]\mu+\mu c^{v}[s,c])^{m}(s+2v)a_{s+v}z^{s+v}}{\frac{v(A-B)}{z^{v}}+\sum_{s=0}^{\infty}(1+[v,c]\mu+\mu c^{v}[s,c])^{m}((s+v)B+Av)a_{s+v}z^{s+v}}\right|$$

$$=\left|\frac{\sum_{s=0}^{\infty}(1+[v,c]\mu+\mu c^{v}[s,c])^{m}(s+2v)a_{s+v}z^{s+v}}{\frac{v(A-B)}{z^{v}}+\sum_{s=0}^{\infty}(1+[v,c]\mu+\mu c^{v}[s,c])^{m}((s+v)B+Av)a_{s+v}z^{s+v}}\right|<1.$$

Since $Re(z) \leq |z|, \forall z(z \in U^{*})$. Define

$$Re\left\{\frac{\sum_{s=0}^{\infty}(1+[v,c]\mu+\mu c^{v}[s,c])^{m}(s+2v)a_{s+v}z^{s+v}}{\frac{v(A-B)}{z^{v}}+\sum_{s=0}^{\infty}(1+[v,c]\mu+\mu c^{v}[s,c])^{m}((s+v)B+Av)a_{s+v}z^{s+v}}\right\}\leq 1. \qquad (7)$$

The value of z we choose on the real axis so that $\delta_{c}D_{\mu,c}^{m}h(z)$ is real.

$$\sum_{s=1}^{\infty}(1+[v,c]\mu+\mu c^{v}[s,c])^{m}(s+2v)a_{s+v}z^{s+v}$$
$$\leq \frac{v(A-B)}{z^{v}}+\sum_{s=1}^{\infty}(1+[v,c]\mu+\mu c^{v}[s,c])^{m}\big((s+v)B+Av\big)a_{s+v}z^{s+v}.$$

Letting $z \to 1^{-}$ through real values,

$$\sum_{s=1}^{\infty}(1+[v,c]\mu+\mu c^{v}[s,c])^{m}(s+2v)a_{s+v}\leq v(A-B)$$
$$+\sum_{s=1}^{\infty}(1+[v,c]\mu+\mu c^{v}[s,c])^{m}\big((s+v)B+Av\big)a_{s+v}.$$

So, we can write (7) as

$$\sum_{s=0}^{\infty}(1+[v,c]\mu+\mu c^{v}[s,v])^{m}[(s+v)(1-B)+P(1-A)]a_{s+v}\leq v(A-B),$$

Sharpness of the outcome now comes after setting

$$h_{s}(z)=z^{-v}+\frac{v(A-B)}{(1+[v,c]\mu+\mu c^{v}[s,c])^{m}[(s+v)(1-B)+P(1-A)]}z^{s+v},\ \ s\geq 0. \blacksquare$$

Corollary (2.2): Let $h \in Lk_{\mu,c}(v,m,A)$. Then

$$a_{s+v}\leq \frac{v(A-B)}{(1+[v,c]\mu+\mu c^{v}[s,c])^{m}[(s+v)(1-B)+P(1-A)]}\ ,\ \ s\geq 0 \qquad (8)$$

The bounds for the growth theorem and distortion theorem for the class $Lk_{\mu,c}(v,m,A)$ are then obtained.

3. Growth and distortion theorems:

Theorem (3.1): If $h \in Lk_{\mu,c}(v,m,A)$, and $(s \geq 1)$, then

$$\frac{1}{r^{v}}-\frac{v(A-B)}{[2v-v(A+B)]}r^{v}\leq\left|\delta_{c}D_{\mu,c}^{m}h(z)\right|$$
$$\leq \frac{1}{r^{v}}+\frac{v(A-B)}{[2v-v(A+B)]}r^{v},(|z|=r<1)\,. \qquad (9)$$

For the function, the outcome is sharp:

$$h_s(z) = z^{-v} + \frac{v(A-B)}{(1+[v,c]\mu+\mu c^v[1,c])^m[(v)(1-B)+P(1-A)]} z. \quad (10)$$

Proof: Let $h \in Lk_{\mu,c}(v,m,A)$. Then by Theorem 1, we get

$$(1 + [v,c]\mu + \mu c^v[1,c])^m[(v)(1-B) + P(1-A)] \sum_{s=0}^{\infty}|a_{s+v}|$$
$$\leq \sum_{s=0}^{\infty}(1 + [v,c]\mu + \mu c^v[s,c])^m[(s+v)(1-B) + P(1-A)]\, a_{s+v}$$
$$\leq v(A-B).$$

Or

$$\sum_{s=0}^{\infty} a_{s+v} \leq \frac{v(A-B)}{(1+[v,c]\mu+\mu c^v[1,c])^m[(v)(1-B)+P(1-A)]}. \quad (11)$$

Hence

$$\left|\delta_c D_{\mu,c}^m h(z)\right| \leq \frac{1}{|z|^v} + \sum_{s=1}^{\infty}(1 + [v,c]\mu + \mu c^v[s,c])^m\, a_{s+v}|z|^{s+v}$$
$$\leq \frac{1}{|z|^v} + (1 + [v,c]\mu + \mu c^v[1,c])^m|z|^v \sum_{s=0}^{\infty} a_{s+v}$$
$$= \frac{1}{r^v} + (1 + [v,c]\mu + \mu c^v[1,c])^m\, r^v \sum_{s=0}^{\infty} a_{s+v}$$
$$\leq \frac{1}{r^v} + \frac{v(A-B)}{[2v-v(A+B)]} r^v. \quad (12)$$

Similarly,

$$\left|\delta_c D_{\mu,c}^m h(z)\right| \geq \frac{1}{|-z|^v} - \sum_{s=0}^{\infty}(1 + [v,c]\mu + \mu c^v[s,c])^m\, a_{s+v}|z|^{s+v}$$
$$\geq \frac{1}{|z|^v} - (1 + [v,c]\mu + \mu c^v[1,c])^m|z|^v \sum_{s=0}^{\infty}|a_{s+v}|$$
$$= \frac{1}{r^v} - (1 + [v,c]\mu + \mu c^v[1,c])^m\, r^v \sum_{s=0}^{\infty}|a_{s+v}|$$
$$\geq \frac{1}{r^v} - \frac{v(A-B)}{[2v-v(A+B)]} r^v. \quad (13)$$

From (12) and (13), we get (9).■

Theorem (3.2): If $h \in Lk_{\mu}, c(v,m,A)$. And $(s \geq 1)$, then

$$\frac{P}{r^{P+1}} - \frac{v^2(A-B)}{[2v-v(A+B)]} r^{v-1}$$
$$\leq \left|\left(\delta_c D_{\mu,c}^m h(z)\right)'\right|$$
$$\leq \frac{P}{r^{P+1}} + \frac{v^2(A-B)}{[2v-v(A+B)]} r^{v-1}, (|z| = r < 1). \quad (14)$$

For the function, the outcome is sharp:

$$h(z) = z^{-v} + \frac{v(A-B)}{(1+[v,c]\mu+\mu c^v[1,c])^m[(v)(1-B)+P(1-A)]} z. \quad (15)$$

proof: Let $h \in Lk_{\mu,c}(v,m,A)$. Then by Theorem (1), we get

$$(1+[v,c]\mu+\mu c^{v}[1,c])^{m}[(v)(1-B)+P(1-A)]\sum_{s=0}^{\infty}|a_{s+v}|$$
$$\leq \sum_{s=0}^{\infty}(1+[v,c]\mu+\mu c^{v}[s,c])^{m}[(s+v)(1-B)+P(1-A)]a_{s+P}$$
$$\leq v(A-B).$$

Or

$$\sum_{s=1}^{\infty} a_{s+v} \leq \frac{v(A-B)}{(1+[v,c]\mu+\mu c^{v}[1,c])^{m}[(v)(1-B)+P(1-A)]} \tag{16}$$

Hence

$$\left|\delta_c D_{\mu,c}^{m} h(z)'\right|$$

$$\leq \frac{v}{|-z|^{v+1}} + \sum_{s=0}^{\infty}(1+[v,c]\mu+\mu c^{v}[s,c])^{m}(s+v)a_{s+v}|z|^{s+v-1}$$
$$\leq \frac{v}{|z|^{v+1}} + (1+[v,c]\mu+\mu c^{v}[s,c])^{m}(s+v)|z|^{v-1}\sum_{s=0}^{\infty} a_{s+v}$$
$$= \frac{v}{r^{v+1}} + (1+[v,c]\mu+\mu c^{v}[s,c])^{m}(v)r^{v-1}\sum_{s=0}^{\infty} a_{s+v}$$
$$\leq \frac{v}{r^{v+1}} + \frac{v^{2}(A-B)}{[2v-v(A+B)]}r^{v-1}. \tag{17}$$

Similarly,

$$\left|\left((\delta_c D_{\mu,c}^{m} h(z)\right)\right|$$

$$\geq \frac{v}{|-z|^{v+1}} - \sum_{s=1}^{\infty}(1+[v,c]\mu+\mu c^{v}[s,c])^{m}(s+v)a_{s+v}|z|^{s+v-1}$$
$$\geq \frac{v}{|z|^{v+1}} - (1+[v,c]\mu+\mu c^{v}[s,c])^{m}|z|^{v-1}\sum_{s=0}^{\infty} a_{s+v}$$
$$= \frac{v}{r^{v+1}} - (1+[v,c]\mu+\mu c^{v}[1,c])^{m}\, r^{v-1}\sum_{s=0}^{\infty} a_{s+v}$$
$$\geq \frac{v}{r^{v+1}} - \frac{P^{2}(A-B)}{[2v-v(A+B)]}r^{v}. \tag{18}$$

From (17) and (18), we get (14). ■

The radii of starlikeness and convexity are fixed in Theorems (4.1) and Theorem (4.2).

4. Radii of starlikeness and convexity theorems:

Theorem (4.1): If $h(z) \in Lk_{\mu,c}(v,m,A)$, then $h(z)$ is meromorphic starlike of order $\theta(0 \leq \theta < P)$ in the disk $|z| < r_1$, where

$$r_1 = ish_s\left\{\frac{(P-\theta)(1+[v,c]\mu+\mu c^{v}[s,c])^{m}[(s+v)(1-B)+P(1-A)]}{(s+3P-\theta)v(A-B)}\right\}^{\frac{1}{s+2P}}, s \geq 1.$$

The result of function $h(z)$ is sharp defined by (6).

Proof: To prove

$$\left|v+\frac{h\prime(z)z}{h(z)}\right| \text{ little than } \; v-\theta \quad hor\ |z| < r_1. \tag{19}$$

But

$$\left|\frac{zh'(z)+vh(z)}{h(z)}\right| = \left|\frac{\sum_{S=1}^{\infty}(s+2v)a_{S+P}z^{S+2v}}{1+\sum_{S=0}^{\infty}a_{S+P}z^{S+2v}}\right|$$

$$\leq \frac{\sum_{S=1}^{\infty}(s+2v)a_{S+P}|z|^{S+2v}}{1-\sum_{S=0}^{\infty}a_{S+P}|z|^{S+2v}}$$

So, (19) will be satisfied if

$$\frac{\sum_{S=0}^{\infty}(s+2v)a_{S+P}|z|^{S+2v}}{1-\sum_{S=1}^{\infty}a_{S+P}|z|^{S+2v}} \leq v-\theta.$$

Or if

$$\sum_{S=1}^{\infty}\frac{(s+3v-\theta)a_{S+v}}{v-\theta}|z|^{S+2v} \leq 1. \qquad (20)$$

Since $h(z) \in Lk_{\mu,c}(v,m,A)$, we have

$$\sum_{S=1}^{\infty}\frac{(1+[v,c]\mu+\mu c^{v}[s,c])^{m}[(s+v)(1-B)+P(1-A)]}{v(A-B)}\ a_{S+P} \leq 1.$$

Hence, (20) is true if

$$\frac{(s+3P-\theta)}{P-\theta}|z|^{S+2P} \leq \frac{(1+[v,c]\mu+\mu c^{v}[s,c])^{m}[(s+v)(1-B)+P(1-A)]}{v(A-B)}$$

Or equivalently

$$|z| \leq \left\{\frac{(P-\theta)(1+[v,c]\mu+\mu c^{v}[s,c])^{m}[(s+v)(1-B)+P(1-A)]}{(s+3P-\theta)v(A-B)}\right\}^{\frac{1}{S+3P}} \quad , s \geq 1.$$

Setting $|z| = r_1$, we get the desired result. ■

Theorem (4.2): If $h(z) \in Lk_{\mu,c}(v,m,A)$, then $h(z)$ is meromorphic function convex of θ order when $(0 \leq \theta < 1)$ in the disk $|z| < r_2$, and

$$r_2 = ish_s\left\{\frac{v(v-\theta)(1+[v,c]\mu+\mu c^{v}[s,c])^{m}[(s+v)(1-B)+P(1-A)]}{(s+v)(s+3v-\theta)v(A-B)}\right\}^{\frac{1}{S+2P}} \geq 1. \qquad (21)$$

The result of function $h(z)$ is sharp is defined by (6).

Proof: It is sufficient to prove that

$$\left|\frac{h''(z)z}{h'(z)}+1+P\right| \leq v-\theta \quad hor\ |z| < r_2. \qquad (22)$$

But this equation

$$\left|\frac{zh''(z)}{h'(z)}+1+v\right|=\left|\frac{zh''(z)+(1+v)h'(z)}{h'(z)}\right|$$

$$\leq\frac{\sum_{s=1}^{\infty}(s+v)(s+2v)a_{s+P}|z|^{s+2v}}{v-\sum_{s=1}^{\infty}(s+v)a_{s+P}|z|^{s+2v}}$$

So, (22) be satisfied if

$$\frac{\sum_{s=1}^{\infty}(s+v)(s+2v)a_{s+P}|z|^{s+2P}}{v-\sum_{s=1}^{\infty}(s+v)a_{s+P}|z|^{s+2P}}\leq v-\theta,$$

or if

$$\sum_{s=1}^{\infty}\frac{(s+v)(s+3v-\theta)a_{s+P}}{v(v-\theta)}|z|^{s+2P}\leq 1. \qquad (23)$$

Since $h(z)\in Lk_{\mu,c}(v,m,A)$, we have

$$\sum_{s=1}^{\infty}\frac{\mathrm{f}(1+[v,c]\mu+\mu c^{v}[s,c])^{m}[(s+v)(1-B)+P(1-A)]}{v(A-B)}\; a_{s+P}\leq 1.$$

Hence, (23) will be true if

$$\frac{(s+v)(s+3v-\theta)}{v(v-\theta)}|z|^{s+2P}\leq\frac{(1+[v,c]\mu+\mu c^{v}[s,c])^{m}[(s+v)(1-B)+P(1-A)]}{v(A-B)},$$

Or equivalently

$$|z|\leq\left\{\frac{v(v-\theta)(1+[v,c]\mu+\mu c^{v}[s,c])^{m}[(s+v)(1-B)+P(1-A)]}{(s+v)(s+3v-\theta)v(A-B)}\right\}^{\frac{1}{s+3P}}\quad ,s\geq 0.$$

Setting $|z|=r_2$, we get the desired result ■

Theorem (4.3): The class $Lk_{\mu,c}(v,m,A)$, is closed for convex linear combinations.

Proof: Let's use the functions

$$h_j(z)=z^{-P}+\sum_{s=1}^{\infty}\left|a_{s+P,j}\right|z^{s+P}\quad ,\quad (j=1,2).$$

Be in the class $Lk_{\mu,c}(v,m,A)$. It is sufficient to show that the function defined by

$$F(z)=(1-t)h_1+th_2\quad ,\quad (0\leq t\leq 1).$$

Is also in the class $Lk_{\mu,c}(v,m,A)$. Since

$$F(z)=\frac{1}{Z^{P}}+\sum_{s=1}^{\infty}\left[(1-t)\left|a_{s+P,1}\right|+t\left|a_{s+P,2}\right|\right]z^{s+P}\ ,(0\leq t\leq 1). \qquad (24)$$

Hence,

$\sum_{s=1}^{\infty}((1+[v,c]\mu+\mu c^{v}[s,c])^{m}\,[(s+v)(1-B)+P(1-A)])\,\,[(1-t)\left|a_{s+P,1}\right|+t\left|a_{s+P,2}\right|]$

$=(1-t)\sum_{s=1}^{\infty}((1+[v,c]\mu+\mu c^{v}[s,c])^{m}\,[(s+v)(1-B)+P(1-A)])\left|a_{s+P,1}\right|$

$+t\,\,\sum_{s=1}^{\infty}((1+[v,c]\mu+\mu c^{v}[s,c])^{m}\,[(s+v)(1-B)+P(1-A)])\left|a_{s+P,2}\right|$

$\leq (1-t)P(A-B)+tP(A-B)=P(A-B).$

Which show that $F(z)\in Lk_{\mu,c}(v,m,A)$ ■

In the subsequent theorem, we get the extreme points of the class $Lk_{\mu,c}(v,m,A)$.

Theorem (4.4): Let $h_0(z)=z^{-P}$ and

$$h(z)=z^{-v}+\frac{P(A-B)}{(1+[v,c]\mu+\mu c^{v}[1,c])^{m}[(s+v)(1-B)+P(1-A)]}z^{s+P},\ \ (s\geq 1).$$

Then $h\in Lk_{\mu,c}(v,m,A),$ if the form may be used:

$$F(z)=\omega_0 h_0(z)+\sum_{s=1}^{\infty}\omega_s h_s(z)\,,\ \ (\omega_s\geq 0,\omega_0+\sum_{s=1}^{\infty}\omega_s=1). \qquad (25)$$

Proof: We impose

$$F(z)=\omega_0 h_0(z)+\sum_{s=1}^{\infty}\omega_s h_s(z)$$

$$h(z)=z^{-v}+\sum_{s=1}^{\infty}\frac{P(A-B)\omega_s}{(1+[v,c]\mu+\mu c^{v}[s,c])^{m}[(s+v)(1-B)+P(1-A)]}z^{s+P}.$$

Then

$$\sum_{s=1}^{\infty}\frac{(1+[v,c]\mu+\mu c^{v}[s,c])^{m}[(s+v)(1-B)+P(1-A)]}{P(A-B)}$$

$$\omega_s\frac{P(A-B)}{(1+[v,c]\mu+\mu c^{v}[s,c])^{m}[(s+v)(1-B)+P(1-A)]}$$

$$=\sum_{s=1}^{\infty}\omega_s=1-\omega_0\leq 1.$$

So, by Theorem (1), $h\in Lk_{\mu,c}(v,m,A)$.

Conversely, we suppose $h\in Lk_{\mu,c}(v,m,A)$. By (7), we have

$$a_{s+P}\leq\frac{P(A-B)}{(1+[v,c]\mu+\mu c^{v}[s,c])^{m}[(s+v)(1-B)+P(1-A)]}\ \ ,s\geq 0.$$

Setting

$$\omega_s=\frac{(1+[v,c]\mu+\mu c^{v}[s,c])^{m}[(s+v)(1-B)+P(1-A)]}{P(A-B)}\,a_{s+P}\,,\ \ s\geq 0.$$

And

$$\omega_0 = 1 - \sum_{s=1}^{\infty} \omega_s .$$

Then

$$h(z) = \omega_0 h_0(z) + \sum_{s=1}^{\infty} \omega_s \, h_s(z).$$

Then

$$h(z) = z^{-v} + \sum_{s=1}^{\infty} a_{s+P} z^{s+P}$$

$$= z^{-P} + \sum_{s=1}^{\infty} \frac{P(A-B)}{(1+[v,c]\mu+\mu c^{v}[s,c])^{m}[(s+v)(1-B)+P(1-A)]} z^{s+P}$$

$$= z^{-P} + \sum_{s=1}^{\infty}(h_s - z^{-1})\omega_s$$

$$= z^{-P}(1 - \sum_{s=1}^{\infty} \omega_s) + \sum_{s=1}^{\infty} \omega_s h_s$$

$$= z^{-P}\omega_0 + \sum_{s=1}^{\infty} \omega_s h_s$$

$$h(z) = \omega_0 h_0(z) + \sum_{s=0}^{\infty} \omega_s h_s(z) \blacksquare$$

Theorem (4.5): Let $h_1(z), h_2(z), \dots, h_\mu(z)$ defined by

$$h_i(z) = z^{-v} + \sum_{s=1}^{\infty} a_{s+v,i} \; z^{s+v}, (a_{s+v,i} \geq 0, i = 1,2, \dots, \mu, s \geq 0) \qquad (26)$$

be a member of the class $Lk_{\mu,c}(v,m,A)$.. Afterward, the mathematical mean of $h_i(z)(i = 1,2, \dots, \mu)$ by means of:

$$f(z) = \frac{1}{\mu} \sum_{i=1}^{\mu} h_i(z) \qquad (27)$$

Is also in the class $Lk_{\mu,c}(v,m,A)$.

Proof: By (26), (27) we can write

$$f(z) = \frac{1}{\mu} \sum_{i=1}^{\mu} (z^{-v} + \sum_{s=1}^{\infty} a_{s+v,i} \; z^{s+v})$$

$$= z^{-v} + \sum_{s=1}^{\infty} (\frac{1}{\mu} \sum_{i=1}^{\mu} a_{s+v,i}) \; z^{s+v}.$$

Since $h_i \in Lk_{\mu,c}(v,m,A)$ for every $(i = 1,2, \dots, \mu)$ therefore, using the Theorem (1), we can prove that

$$\sum_{s=1}^{\infty}(1 + [v,c]\mu + \mu c^{v}[s,c])^{m}[(s+v)(1-B) + P(1-A)](\frac{1}{\mu} \sum_{i=1}^{\mu} a_{s+v,i})$$

$$= \frac{1}{\mu} \sum_{i=1}^{\mu} (\sum_{s=1}^{\infty} (1 + [v,c]\mu + \mu c^{v}[s,v])^{m}[(s+v)(1-B) + P(1-A)] a_{s+v,i})$$

$$\leq \frac{1}{\mu} \sum_{i=1}^{\mu} v(A-B) = v(A-B). \blacksquare$$

Conclusion

In our present paper, we were essentially motivated by the recent research going on in this field of study and we have first introduced a class of meromorphic multivalent function with q-differential operator. We next investigate some useful properties such as coefficient estimates, growth and distortion theorems, radii of starlikeness and convexity, convex linear combination, extreme points theorem and the arithmetic mean for the present subclass of meromorphic multivalent functions.

Finally, we would like to explain that the new in this study that is some geometric properties of meromorphic multivalent functions for w-differential operator is generalized on starlikeness or convex functions.

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