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Complex Valued Rectangular $Ç_b$- Metric spaces Topological Properties and Application

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Abstract

We will begin by introducing the concept of complex valued rectangular $Ç_b$- metric spaces which generalised the notion of rectangular metric space and complex valued b -metric space. Then, some metric and topological properties, fixed point results and examples connected with certain contractions are obtained in the setting of complex valued rectangular Gb -metric spaces. Also, we will enhance the main result with application, where, unique common solution of the system of Urysohn integral equation is given. The axis in which the topological properties rotate is wide and branching, and it difficult to confine it to a small research, but in this paper we will try to extend with the complex valued rectangular $Ç_b$-metric spaces. In addition to presenting some examples that give a somewhat clear perception of the concept

Introduction

In the world of mathematics, a theory emerged [1] that is considered one of the most effective theoretical tools, and it is the fixed-point theory that was adopted by the creative scientist Banach in the middle of the last century [2–4]. This theory has a lengthy history and has undergone substantial research in a variety of fields, including economy, area programming, artificial intelligence, control optimization and chaos, mentioned in the references list [5–7]. We used to use this theory to achieve the existence of a solution to many different equations, such as differential equations, partial differential equations, and integral equations, in addition to proving the unity of solution [8]. Among these equations, the science of optics and electromagnetic waves invaded the world in an exaggerated way so many academics generalised metric spaces and theoretical and applied results [9] as indicated in [10–12] Here, light is highlighted on the generalisation of complex value metrics [13].

In 2011 [14], the researcher Azam and others presented the idea of metric spaces of complex value, so that this idea developed many results on the fixed point and provided logical justifications for contraction [15], and highlights the importance of the possibility of applying complex valued metric spaces in many fields in mathematics and most of its fields. And physics and justifying its applications by mathematical theories Including mechanical engineering, algebraic engineering, applied mathematics, electrical engineering, thermodynamics and hydrodynamics. Azam's point of view, in which he introduced the concept of well-known metric spaces, was more general and works to show the fixed point theorems in a more generalised manner in the rest of the sciences that this theory serves and adopts the proof of the facts proposed by researchers in [16] various sciences. The idea was developed for b-metric spaces by Rao and a number of researchers.

In 2013 it showed a lot of fixed points that are consistent with the common fixed points. Recently Ozgur [17] presented another concept, which is the complex valued rectangular b-metric space and many concepts of contraction appeared in this space, as Ozgur presented two concepts of contraction in the new space [19], and then Mustafa and Sims worked on Generalized metric space structure and named after Ç -metric spaces [20]. Aghajani et al. presented a new generalization of metric space recently In order to combine the two spaces, b-metric spaces and Ç -metric space. It is possible to extract a number of results and previous works in the field of the metric fixed point to obtain new results, (as presented in [21]).

Throughout this paper, we will discuss the idea of generalization of the rectangular $Ç_b$-metric space in complex valued space and present some topological properties and fixed point theorems.

Complex Valued Rectangular $Ç_b$- Metric space

Let C is collection of complex numbers then$\delta_1, \delta_2 \in C$ defines the partial order relationship $\precsim$ on C as follows

$\delta_1 \precsim \delta_2$ if and only if $\text{Re}(\delta_1) \leq \text{Re}(\delta_2)$ and $\text{Im}(\delta_1) \leq \text{Im}(\delta_2)$

that is $\delta_1 \precsim \delta_2$ if one of these following condition is holds

(i) $Re(\delta_1) = Re(\delta_2)$ and $Im(\delta) = Im(\delta_2)$;

(ii) $Re(\delta_1) < Re(\delta_2)$ and $Im(\delta_1) = Im(\delta_2)$;

(iii) $Re(\delta_1) = Re(\delta_2)$ and $Im(\delta_1) < Im(\delta_2)$;

(iv) $Re(\delta_1) < Re(\delta_2)$ and $Im(\delta_1) < Im(\delta_2)$.

In particular, the symbol $\delta_1 \precnsim \delta_2$ mean that $\delta_1 \neq \delta_2$ and one of (i-iv) is satisfied and we will write $\delta_1 \prec \delta_2$ if only (iv) is satisfied.

Remark 2.1 [17] the following terms satisfies

(i) $a, b \in R$ and $a \leq b \Rightarrow a\delta \precsim b\delta, \ \forall \ \delta \in C$,

(ii) $0 \precsim \delta_1 \precnsim \delta_2 \Rightarrow |\delta_1| < |\delta_2|$,

(iii) $\delta_1 \precsim \delta_2$ and $\delta_2 \prec \delta_3 \Rightarrow \delta_1 \prec \delta_3$.

As a new approach of Azam's space [18] the following definition may be proper

Definition 2.2. For X which is a non-empty set and $s \geq 1$, if a complex valued function Ç on X^3satisfies the conditional below:

(i) $Ç(x, v, h) = 0$, if $x = v = h$,

(ii) $0 \prec Ç(x, x, v)$ for all $x, v \in X$ with $x \neq v$,

(iii) $Ç(x, x, v) \precsim Ç(x, v, h)$ for all $x, v, h \in X$, with $v \neq h$,

(iv) $Ç(x,v,h) = Ç(x,v,h) = Ç(v,x,h) = \cdot\cdot$ (All three variables are Symmetric for every $x,w,h \in X$, and every distinct two points $\vartheta,\alpha \in X\backslash\{x,v,h\}$. Therefore $(X,Ç)$ is refe to a complex rectangular $Ç_b$–metric space, shortly, CRÇ$_bM$-space.

We should see, if $s = 1$ then $(X,Ç)$ is called a complex rectangle Ç- metric space (CRÇ-space), if $\vartheta = \alpha$ then $(X,Ç)$ be complex $Ç_b$-MS ($CÇ_b$- metric space); and if $s = 1$ and $\vartheta = \alpha$ then it is said complex Ç-space (CÇ-space)

Example 2.3. For any $X = E \cup \{1,2\} \cup B$, such that $E = \left\{\frac{1}{y}\ ; y = 2,3,4,\ldots\right\}$ and $B = \{3,4,5,\ldots\}$,

Define $Ç : X \times X \times X \to C$ as

$$Ç(x,v,h) = Ç(x,h,v) = Ç(v,x,h) = \cdots, \text{ for all } x,v,h \in X$$

and

$$Ç(x,v,h) = \{0 \quad if\ \ x = v = h; \qquad 4j \quad if\ \ x,v,h \in E\ and\ x \neq w \neq h; \quad \frac{j}{2y} \quad if\ x \in E\ \ and\ \ v \neq h \in \{1,2\}; \quad j \quad otherwise \qquad , \qquad \ldots(1)$$

Since for every $x,v,h \in X$ and non-negative distinct points $\vartheta,\alpha \in X\backslash\{x,v,h\}$, and $j > 0$ is constant.

Now, check conditions (i-v) in Definition (2.2)

(i) If $Ç(x,v,h) = 0$ then $x = v = h$, from (1).
Therefore

$$Ç(x,v,h) = Ç(x,h,v) = Ç(v,x,h) = \cdots = 0, \text{ for all } x,v,h \in X.$$

(ii) Assumes that $x \neq v \neq h$. Then $Ç(x,v,h) = 4j \succ 0$, for every $j > 0$.

(iii) Third condition hold $Ç(x,x,v) \precsim Ç(x,v,h)$,
because $Ç(x,x,v) = j \precsim 4j = Ç(x,v,h)$, for all $x,v,h \in X$, with $v \neq h$.

(iv) Its easy show that from definition.

(v) To prove pentagonal inequality let for every $x,v,h \in X$ and non-negative distinct points $\vartheta,\alpha \in X\backslash\{x,v,h\}$, and $j > 0$ is constant, there is $s > 1$, therefore

$$Ç(x,v,h) \precsim s\big(Ç\,(x,\vartheta,\vartheta) + Ç\,(\vartheta,\alpha,\alpha\,) + Ç\,(\alpha,v,h)\big)$$

were

$$Ç(x,v,h) = 4j \precsim s(j+j+j) = 6j, \text{ with } s = 2 > 1,\ \forall\, \vartheta,\alpha \in X\backslash\{x,v,h\}.$$

Then $(X,Ç)$ is CRÇ$_b$MS with coefficient $s = 2 > 1$.

However, $(X\ ,Ç)$ is not a CRÇM space, because

$$Ç\left(\frac{1}{2},\frac{1}{4},\frac{1}{7}\right) = 4j \succ \frac{9j}{2y} = Ç\left(\frac{1}{2},2,2\right) + Ç(2,2,1) + Ç\left(1,\frac{1}{4},\frac{1}{7}\right).$$

and is not $CÇ_b$MS as there does not exist $s > 1$ such that
$Ç(x,v,h) \precsim s[Ç(x,\vartheta,\vartheta) + Ç(\vartheta,v,h)]$, for all $x,v,h,\vartheta \in X$.

(v) $Ç(x,v,h) \precsim s\big(Ç\,(x,\vartheta,\vartheta) + Ç\,(\vartheta,\alpha,\alpha) + Ç\,(\alpha,v,h)\big)$, (Pentagonal inequality).

Example 2.5. Consider a function $Ç: X^3 \to C$ on a nonempty set X, which a complex Ç–metric space by

$$Ç(x,v,h) = [Ç(x,v,h)]^r, \text{ for all } x,v,h \in X \text{ and } r \geq 1.$$

Then, $(X, Ç)$ is a R$Ç_b$M space with $s = 3^{r-1}$. Shows us that, Ç is agrees to the terms (i) – (iv) of the Definition (2.2). After that, the convexity of $f(x) = x^r$ for $x \geq 0$, implies (v) as follows:

$$Ç(x, v, h) = (b - h|)^r \preccurlyeq 3^{r-1}[(|x - b| + |x - b|)^r + (|b - n| + |o - n|)^r + (n - h|)^r$$
$$= 3^{r-1}[Ç(x, b, b) + Ç(b, n, n) + Ç(n, v, h)].$$

Hence, $(X, Ç)$ is a CR$Ç_b$MS with $s = 3^{r-1}$, for all $b, n \in X \setminus \{x, v, h\}$

Example 2.6. Let $X = R$ and we define $Ç : X^3 \to C$, as

$$Ç(x, v, h) = (|x - v| + |v - h| + |x - h|)^r + i(|x - v| + |v - h| + |x - h|)^r$$

for every different element $x, v, h \in X$ and non-negative $I, l \in X \setminus \{x, v, h\}$, and all $(r \geq 1)$

(i) If $Ç(x, v, h) = 0$ then $(|x - v| + |v - h| + |x - h|)^r = 0$,

and $i(|x - v| + |v - h| + |x - h|)^r = 0$.

Hence $x = v = h$. Therefore

$Ç(x, v, h) = (|x - v| + |v - h| + |x - h|)^r + i(|x - v| + |v - h| + |x - h|)^r = 0$

(ii) Assumes that $x \neq v \neq h$. Then

$$Ç(x, v, h) = (|x - v| + |v - h| + |x - h|)^r + i(|x - v| + |v - h| + |x - h|)^r$$
$$\succcurlyeq |x - v|^r + i|x - v|^r \succ 0,$$

(iii) Since $|x - v| \preccurlyeq |x - h| + |v - h|$,

we have

$Ç(x, x, v) \preccurlyeq (|x - v| + |v - h| + |x - h|)^r + i(|x - v| + |v - h| + |x - h|)^r = Ç(x, v, h)$.

(iv) Clearly, $(x, v, h) = Ç(Q\{x, v, h\})$, where Q permutation of x, v, h.

(v) $Ç(x, v, h) = (|x - v| + |v - h| + |x - h|)^r + i(|x - v| + |v - h| + |x - h|)^r$

$$\preccurlyeq 3(|x - \tau| + |\tau - \tau| + |\tau - \gamma| + |\gamma - \gamma|)^r + 3^{r-1} i(|x - \tau|$$
$$+ |\tau - \tau| + |\tau - \gamma| + |\gamma - \gamma|^r + (\gamma - h|)^r + 3^{r-1} i(\gamma - h|)^r$$
$$= 3\big(Ç(x, \tau, \tau) + Ç(\tau, \gamma, \gamma) + Ç(\gamma, v, h)\big).$$

Hence, $(X, Ç)$ is a R$Ç_b$M space with $s = 3^{r-1}$ but it is not a Ç -metric on C indeed by taking $x = 3$, $v = 5$, $h = 7$ *and* $\tau = 4$, $\gamma = 6$, we have

$$Ç(3,5,7) = \frac{1}{5}(|3 - 5| + |5 - 7| + |3 - 7|)^2 = \frac{64}{5} \succ \frac{36}{5} = Ç(x, \tau, \tau) + Ç(\tau, \gamma, \gamma) + Ç(\gamma, v, h)$$
$$= \frac{4}{5} + \frac{16}{5} + \frac{16}{5}.$$

Example 2.7. The function defines by

$$Ç_b(x, v, h) = max\{|x - v|, |v - h|, |x - h|\}^r + imax\{|x - v|, |v - h|, |x - h|\}^r$$

On $X = [0,1)$, for every $x, h, v \in X$ and non-negative distinct points $L, O \in X \setminus \{x, v, h\}$, and all $(r \in N)$

(i) If $Ç(x, v, h) = 0$, then,

$$max\{|x - v|, |v - h|, |x - h|\}^r + imax\{|x - v|, |v - h|, |x - h|\}^r = 0,$$

Then

$$max\{|x - v|, |v - h|, |x - h|\}^r = 0 \text{ and } imax\{|x - v|, |v - h|, |x - h|\}^r = 0,$$

Hence $x = v = h$.

Therefore

$$Ç(x, v, h) = max\{|x - v|, |v - h|, |x - h|\}^r + imax\{|x - v|, |v - h|, |x - h|\}^r = 0.$$

(ii) Suppose that $x \neq v \neq h$.

Then

$$Ç(x, v, h) = max\{|x - v|, |v - h|, |x - h|\}^r + imax\{|x - v|, |v - h|, |x - h|\}^r$$
$$\succcurlyeq |x - v|^r + i|x - v|^r \succ 0,$$

(iii) Since $|x - v| \preccurlyeq |x - h| + |v - h|$ we have

$$Ç(x, x, v) = max(|x - x|, |x - v|, |x - v|)^r + imax(|x - x|, |x - v|, |x - v|)^r$$

$$= |x-v|^r + i|x-v|^r$$

$$\preccurlyeq max(|x-v|,|v-h|,|x-h|)^r + imax(|x-v|,|v-h|,|x-h|)^r$$
$$= Ç(x,w,h).$$

(iv) It is easy show that $(x,v,h) = Ç(Q\{x,v,h\})$, where Q permutation of x,v,h.

(v) Lemma (2.4) implies that

$$Ç(x,v,h\,) = max(|x-v|,|v-h|,|x-h|)^r + imax(|x-v|,|v-h|,|x-h|)^r$$
$$\preccurlyeq\ 3^{r-1}(L-h|)^r + i3^{r-1}(L-h|)^r$$
$$= 3\big(\,Ç(x,q,q) + Ç(q,o,o) +\ Ç(o,v,h)\big), \text{for all } q,o \in X\setminus\{x,v,h\}$$

Hence, $(X,Ç)$ is a $RÇ_b$MS with $s = 3^{r-1}$ but $(X,Ç)$ is not a complex valued RÇ- metric spaces when $r \neq 1$, $x = \frac{1}{2}$, $v = \frac{1}{7}, h = \frac{1}{4}$, and for $q = \frac{1}{3}$ and $o = \frac{1}{12}$, with $r = 2$.

$$Ç(x,v,h) = max(|x-v|,|\,v-h|,|x-h|)^2 + imax(|x-v|,|v-h|,|x-h|)^2$$
$$= \frac{25}{196} + i\frac{25}{196} \succ \frac{17}{144} + i\frac{17}{144} = \big(\,Ç(x,q,q) + Ç(q,o,o) + Ç(o,v,h)\big).$$

Proposition 2. 6. Let $(X,Ç)$ be a $RÇ_b$MS and $\varepsilon > 0$, then for each $x,v,h \in X$, we have:

(1) if $Ç(x,v,h) = 0$, then $x = v = h$;

(2) $Ç_b(x,v,h) \precsim s(Ç_b(x,x,v) + Ç_b(x,x,h)$, for all $x,v,h \in X$,

(3) $Ç_b(x,v,v) \precsim 2s\big(Ç_b(v,x,x)\big)$;

(4) $Ç_b(x,v,h) \precsim s\big(Ç_b(x,a,v) + Ç_b(a,v,h)\big)$,

(5) if $Ç(x_r,x_t,x_l) \prec \varepsilon$, for any $r,t,l \in N$, then $Ç(x_r,x_t,x_t) \prec \varepsilon$ and $Ç(x_r,x_r,x_l) \prec \varepsilon$.

In a usual manner, can define a rectangular b-metric space (RbM-space) by coefficient $s \geq 1$ induced by $RÇ_b$MS as below. Firstly, recalling the definition of RbM-space [21].

Definition 2.7. A function $d: X \times X \to C$ is called RbM-space for a non-empty set X with a coefficient $s \geq 1$ if:

(1) $d(x,v) = 0 \Leftrightarrow x = v$,

(2) $d(x,v) = d(v,x)$,

(3) $\exists$ a real number $s \geq 1$ satesfy

$$d(x,v) \precsim s[d(x,c) + d(c,b) + d(b,v)]$$

for any different $x,v \in X$, and all $c,b \in X\setminus\{x,v\}$.

Proposition 2.8. Let $(X,Ç)$ be a $RÇ_b$MS. then the subsequent operation $dÇ$ *on* X^2

$$dÇ(x,v) = Ç(x,v,v) + Ç(v,x,x),$$

defines a RbM-space on X.

Proof: We will try to prove the conditions included in the definition 2.7 are fulfilled for $dÇ(x,v)$.

(1) If $dÇ(x,v) = 0$, then $Ç(x,v,v) + Ç(v,x,x) = 0$ and according to Proposition 2.6, we have $x = v$.

(2) If $x = v$, then $Ç(x,v,v) + Ç(v,x,x) = 0$, and $dÇ(x,v)$ =0,

(3) Property (iv) of the Definition (2.2) implies that

$$dÇ(x,v) = Ç(x,v,v\,) + Ç(v,x,x) = Ç(\,v,x,x) + Ç(x,v,v) = dÇ(v,x).$$

(4) By (v) of the Definition (2.2), for distinct points $\varphi, e \in X\setminus\{x,v\}$, it follows that

$$dÇ(x,v) = Ç(x,v,v) + Ç(v,x,x)$$
$$\precsim s[\,Ç(x,\varphi,\varphi) + Ç(\varphi,e,e) + Ç(e,v,v)] + s[\,Ç(v,e,e) + Ç(e,\varphi,\varphi) + Ç(\varphi,x,x)]$$
$$= s[Ç(x,\varphi,\varphi) + Ç(\varphi,x,x) + Ç(\varphi,e,e) + Ç(e,\varphi,\varphi) + Ç(e,v,v) + Ç(v,e,e)]$$
$$= s[dÇ(x,\varphi) + dÇ(\varphi,e) + dÇ(b,v)].$$

Definition 2.9. $RÇ_b$MS is said to be symmetric if

$$Ç(x,v,v) = Ç(x,x,v), \quad \text{for all } x,v \in X .$$

Remark 2.10. The next example about rectangular $Ç_b$-metric Ç is symmetric if Ç –metric Ç* is symmetric.

We will use the following example

Example 2.11. [10] Let $X = \{\alpha, \rho\}$, let, $Ç_b(\alpha,\alpha,\alpha) = Ç_b(\rho,\rho,\rho) = 1$, $Ç_b(\alpha,\rho,\rho) = 2$, and extend $Ç_b$ for all of $X \times X \times X$ by the symmetry property of the variables. Hence it easy expect that $Ç_b$ is a $Ç_b$-metric but $Ç_b(\alpha,\rho,\rho) \neq Ç_b(\alpha,\alpha,\rho)$

Therefore, if $RÇ_bM$ –space is symmetric, then $RÇ_b$MS is "topologically equivalent" to a CR_bM-space.

3. Topological Properties of CR$Ç_b$M-spaces

Firstly, begin with

Definition 3.1. Let $(X , Ç)$ be a R$Ç_b$MS, for $x_0 \in X$, $0 < u \in C$ then Ç-ball with radius r and center x_0 and is $B_{Ç_b}(x_0,u) = \{v \in X: Ç(x_0,v,v) \prec u\}$.

Remark 3.2. the balls $B_{Ç_b}$ which open in R$Ç_b$M space is not necessarily open set. In example (2.3),

$$B_{Ç_b}\left(\frac{1}{2},\frac{n}{2}\right) = \left\{\frac{1}{2},1,2\right\}, \qquad \text{for } n > 0,$$

and there is no open ball with centre 1 and contained in $B_{Ç_b}\left(\frac{1}{2},\frac{n}{2}\right)$.

So that $B_{Ç_b}\left(\frac{1}{2},\frac{n}{2}\right)$ cannot be an open set.

Proposition 3.3. Let $(X,Ç)$ be a $CRÇ_b$- space, then for any, $x_0 \in X$, $u > 0$,, have,

(1) If $Ç_b(x_0,x,v) \precsim u$, then $x,v \in B_{Ç_b}(x_0,u)$,

(2) If $v \in B_{Ç_b}(x_0,u)$ then there exists a $0 < \gamma \in C$ such that $B_{Ç_b}(j,\gamma) \subseteq B(x_0,u)$.

(3) The family $B = \{B_{Ç_b}(x,u) : x \in X, u > 0\}$ constructs with the base of the topology on X.

Proof: The first part is getting from (Definition 2.2 (iii)), while (2) follows from (v) with $\gamma = u - Ç_b(x_0,v,v)$. Therefore, this implies (3).

Proposition 3.4. Let $(X , Ç)$ be a $CRÇ_b$-MS therefore for any, $x_0 \in X$, $u > 0$, we have,

$$B_{Ç_b}(x_0,\frac{1}{3}u) \subseteq B_{dÇ_b}(x_0,u) \subseteq B_{Ç_b}(x_0,u).$$

Remark 3.5. It should be noted that Hausdorffness property for CR_bM space may be not available always, such as, see the following example:

Example 3.6. Let (X,d_b) be a CR_bM space with $s = 2$, define $Ç: X \times X \times X \to C$ by the formula

$$Ç(x,v,h) = \{d_b(x,v), d_b(x,h), d_b(h,x)\},$$

obviously, Ç satisfies conditions (i)-(iv) of the (Definition (2.2)), so we only need to verify that (v) is hold. Indeed.

(1) if $x \neq v \neq h$, then $Ç(x,v,h) = \{d_b(x,v), d_b(x,h), d_b(h,x)\}$.

(2) if $x = v = h$, then $Ç(x,v,h) = Ç(x,x,h) = d_b(x,h)$, for distinct points $a,b \in X \setminus \{x,v\}$, we have

$$Ç(x,v,h) \precsim s[d_b(x,a) + d_b(a,b) + s\{d_b(b,v), d_b(v,h), d_b(h,b)\}$$
$$= s[Ç(x,a,a) + Ç(a,b,b) + Ç(b,v,h)].$$

Hence, $(X,Ç)$ be a $CRÇ_b$- space with $s = 2$. However, there does not exist any $r_1, r_2 > 0$, such that

$$B_{Ç_b}(2,r_1) \cap B_{Ç_b}(3,r_2) = \emptyset.$$

Thus $(X,Ç)$ is not Hausdorff.

Definition 3.7. Let $(X\ ,Ç)$ be a $CRÇ_b$- space, $\{x_r\}\subseteq X.$ and $x\in X$. Then

(i) A sequence $\{x_r\}$ is convergent to some x if for every $\epsilon>0,\ \exists\ r_0\in N$ such that $Ç(x,x_r,x_t)\prec\epsilon$ for all $r,t>r_0$ and it is indicated by $x_r\to x$ as $r\ \to\ \infty$.

(ii) A sequence$\{x_r\}$ is "Cauchy sequence" if for every $\epsilon>0\ \exists\ r_0\in N$ such that

$$Ç(x_r,x_t,x_l)\prec\epsilon, \text{ for all } r,t,l>r_0.$$

(iii) If any $Ç_b$-Cauchy sequence in X is $Ç_b$-convergent to some element in X, the CR$Ç_b$M-space $(X,Ç)$ is said to be complete.

Definition 3.8. If $(X,Ç_b)$ and $(X',Ç'_b)$are two $CRÇ_b$-metric spaces, so a function $f{:}\,(X,Ç_b)\to(X',Ç'_b)$, then and only then, is it claimed that f is $Ç_b$-continuous to some point in X if and only if for any $\epsilon>0,\ \exists\ \partial>0$ and $x,v\in X$, such that

$$Ç_b(a,x,v)\prec\ \partial, \text{ implies } Ç'_b\big(f(a),f(x),f(v)\big)\prec\epsilon.$$

Since metric topologies are $Ç_b$-metric topologies, so we obtain:

Proposition 3.9. The function $f{:}\,X\to X'$ is $Ç_b$ –continuous at a point x on X if and only if it is $Ç_b$-sequentially continuous at some x, assuming that $(X\ ,Ç_b)\ ,(X',Ç'_b)\ are$ $Ç_b$-metric spaces.

Proof: we can follow original proof in general metric space so we omit it.

Definition 3.10. Let $(X,Ç_b)$ is a $CRÇ_b$-metric spaces, then the function $Ç_b(x,v,h)$ is collectively continuous in each of its three variables.

Given the significance of these two lemmas and the fact that they are counterparts in [22], we express them here for rectangular complex valued $Ç_b$-metric spaces.

Lemma 3.12. Let $(X,Ç)$ be a R$Ç_b$MS, and $\{x_r\}$ is any sequence in X. Then this sequence $\{x_r\}$to be converges to some point x in X, $\Leftrightarrow$ $|\ Ç(x,x_r,x_t)|\to 0$, as $r,t\to\ \infty$.

Lemma 3.13. For a complex R$Ç_b$MS denoted by $(\ X,Ç)$ if $\{x_r\}$ is a sequence in X. Then $\{x_r\}$to be a $Ç_b$-Cauchy sequence in X, $\Leftrightarrow$ $|Ç(x_r,x_t,x_l)\ |\to 0$ as $r\to\ \infty$.

4. Application to integral equations

Throughout this section, $(X,Ç)$ will be Hausdorff and $Ç$ will be continuous in three variables. Firstly, recall Urysohn's lemma.

Lemma 4.1[15]. A topology space (X,τ) is considered a normal space. if and only if a function $f{:}\,X\to[0,1]$ closed sets for all F_1 and F_2 of X, such that $f(F_1)=0$, or $f(F_2)=1$.

Also define the quadratic Urysohn integral equation in the form

$$x(\zeta)=a(\zeta)+f\big(\zeta,x(\zeta)\big)\int_0^\infty u\big(\zeta,s,x(s)\big)ds\ ,\ \ \zeta\in R^+=[0,\infty), \quad \dots(5)$$

Additionally, integral equations have specific instances. The form of the Urysohn integral equation on bounded interval

$$x(\zeta)=a(\zeta)+\int_0^T u\big(Q,s,x(s)\big)ds,\ \ \zeta\in R^+,$$

and the Urysohn integral equation on unbounded interval

$$x(\zeta)=a(\zeta)+\int_0^\infty u\big(\zeta,s,x(s)\big)ds,\ \ \zeta\in R^+.$$

Are special cases of the equation (1), among others, the integral equation in (5) has a continuous and bounded solution on R^+ which vanishes at infinity.

Definition 4.2. Let S and T be mappings from a CRÇ$_b$MS $(X, Ç)$ into itself. The mappings S and are said to be compatible if $Ç(STx_r, TSx_r) = 0$, whenever $\{x_r\}$is a sequence in X, such that

$$\lim_{r\to\infty} Tx_r = \lim_{n\to\infty} Sx_r = c \text{ , for some } c \in X.$$

We will present the following main theorem, which gives the possibility of application in another field, as will be discussed later.

Theorem 4.3. Let $(X, Ç_b)$ be complete CRÇ$_b$MS and $S, T, W, \psi, R, \gamma: X \times X \times X \to X$ be self mappings hold the conditions:

(i) $S(X) \subset \psi(X)$, $T(X) \subset R(X)$ and $W(X) \subset \gamma(X)$.

(ii) $d(S_x, T_v, W_h) \precsim \frac{\lambda}{s^2} Ç(x, v, h)$, if $s > 1$ and $\lambda \in (0,1)$ for all $x, v, h \in X$, where

$$Ç(x, v, h) = \{d(\psi_x, R_v, \gamma_h), d(\psi_x, S_x, S_x), d(R_v, T_v, T_v), d(\zeta_h, W_h, W_h),$$
$$\frac{1}{3}[d(\gamma_h, S_x, T_v) + d(R_v, T_v, W_h) + d(\psi_x, S_x, W_h) + d(\gamma_h, W_h, T_v) + d(\psi_x, S_x, T_v)],$$
$$\frac{d(\psi_x, S_x, S_x), d(R_v, T_v, T_v), d(\gamma_h, W_h, W_h)}{1 + d(\psi_x, R_v, \gamma_h)}\},$$

(iii) (S, R, γ), is compatible and $(T, \psi, \gamma)$and (W, ψ, R) is weakly compatible,

(iv) Either S or T or W is continuous.

Then S, T, W, ψ, R and γ have unique common fixed point in X.

Our first new results in this section are the following Urysohn-type integral equation. Consider the system

$$x(\zeta) = \int_{\dot{o}}^{\tilde{\omega}} U_1(\zeta, s, x(\zeta))ds + f(\zeta)\, x(\zeta)$$
$$= \int_{\dot{o}}^{\tilde{\omega}} U_2(\zeta, s, x(\zeta))ds + k(\zeta)\, x(\zeta)$$
$$= \int_{\dot{o}}^{\tilde{\omega}} U_3(\zeta, s, x(\zeta))ds + p(\zeta). \qquad \text{... (6)}$$

where $f(\zeta), p(\zeta)$ and $k(\zeta)$ are unknown variables for each $\zeta \in [\dot{o}, \tilde{\omega}]$, $\dot{o} > 0$,
$x(\zeta)$ is the deterministic free term defined for $\zeta \in [\dot{o}, \tilde{\omega}]$,
$U_1(\zeta, s), U_2(\zeta, s)$ and $U_3(\zeta, s)$ are deterministic kernels defined for $\zeta, s \in [\dot{o}, \tilde{\omega}]$.

Theorem 4.4. For $(X$,Ç$)$ which be a complete C RÇ$_b$MS there are parameter $s \geq 1$ and $S, T, R: X \times X \times X \to$ C be mappings. If there are three mappings $\alpha, P, L, J: C_+ \to [0, 1)$ considers all these elements $x, v, h, u, s, p, c, j, d \in X$, we get

(i) $\alpha(x) + P(x) + L(x) + J(x) \prec 1$,

(ii) $Ç(S(x, v, h), T(u, s, p), R(c, j, d))$

$$\preccurlyeq \alpha \frac{Ç(x, u, c), Ç(v, s, j), Ç(h, p, d)}{3} + P \frac{Ç(S(x, v, h), T(u, s, p), R(c, j, d)), Ç(x, u, c)}{1 + Ç(x, u, c), Ç(v, s, j), Ç(h, p, d)}$$
$$+ L \frac{Ç(S(x, v, h), T(u, s, p), R(c, j, d)), Ç(j, v, d)}{1 + Ç(x, u, c), Ç(v, s, j), Ç(h, p, d)}$$
$$+ J \frac{Ç(S(x, v, h), T(u, s, p), R(c, j, d)), Ç(h, v, k)}{1 + Ç(x, u, c), Ç(v, s, j), Ç(h, p, d)}.$$

Then, in X, S, T and R have tripled common fixed point which are unique.

Proof: For x_0, v_0 *and* h_0 represent any location within X. We create the sequence $\{x_r\}$, $\{v_r\}$ and $\{h_r\}$ in X such that

$x_{3r+1} = S(x_{3r}, v_{3r}, h_{3r})$, $\quad v_{3r+1} = S(v_{3r}, x_{3r}, h_{3r})$, $\quad h_{3r+1} = S(h_{3r}, v_{3r}, x_{3r})$,

$$x_{3r+2} = T(x_{3r+1}, v_{3r+1}, h_{3r+1}), \quad v_{3r+2} = T(v_{3r+1}, x_{3r+1}, h_{3r+1}),$$
$$h_{3r+2} = T(h_{3r+1}, v_{3r+1}, x_{3r+1})$$
$$x_{3r+3} = R(x_{3r+2}, v_{3r+2}, h_{3r+2}), \quad v_{3r+3} = R(v_{3r+2}, x_{3r+2}, h_{3r+2}),$$
$$h_{3r+3} = R(h_{3r+2}, v_{3r+2}, x_{3r+2}), \text{ for all } r \geq 0.$$

Then, for all $r \geq 0$, we have

$$Ç(x_{3r+1}, x_{3r+2}, x_{3r+3}) = Ç(S(x_{3r}, v_{3r}, h_{3r}), T(x_{3r+1}, v_{3r+1}, h_{3r+1}), R(x_{3r+2}, v_{3r+2}, h_{3r+2}))$$
$$\preccurlyeq \alpha \frac{Ç(x_{3r}, x_{3r+1}, x_{3r+2}), Ç(v_{3r}, v_{3r+1}, v_{3r+2}), Ç(h_{3r}, h_{3r+1}, h_{3r+2})}{3}$$
$$+P \frac{Ç(S(x_{3r}, v_{3r}, h_{3r}), T(x_{3r+1}, v_{3r+1}, h_{3r+1}), R(x_{3r+2}, v_{3r+2}, h_{3r+2}))Ç(x_{3r}, x_{3r+1}, x_{3r+2})}{1 + Ç(x_{3r}, x_{3r+1}, x_{3r+2}) + Ç(v_{3r}, v_{3r+1}, v_{3r+2}) + Ç(h_{3r}, h_{3r+1}, h_{3r+2})}$$
$$+ L \frac{Ç(S(x_{3r}, v_{3r}, h_{3r}), T(x_{3r+1}, v_{3r+1}, h_{3r+1}), R(x_{3r+2}, v_{3r+2}, h_{3r+2}))Ç(v_{3r}, v_{3r+1}, v_{3r+2})}{1 + Ç(x_{3r}, x_{3r+1}, x_{3r+2}) + Ç(v_{3r}, v_{3r+1}, v_{3r+2}) + Ç(h_{3r}, h_{3r+1}, h_{3r+2})}$$
$$+ J \frac{Ç(S(x_{3r}, v_{3r}, h_{3r}), T(x_{3r+1}, v_{3r+1}, h_{3r+1}), R(x_{3r+2}, v_{3r+2}, h_{3r+2}))Ç(h_{3r}, h_{3r+1}, h_{3r+2})}{1 + Ç(x_{3r}, x_{3r+1}, x_{3r+2}) + Ç(v_{3r}, v_{3r+1}, v_{3r+2}) + Ç(h_{3r}, h_{3r+1}, h_{3r+2})}$$

Implies that

$$(1 - P - L - J)Ç(x_{3r+1}, x_{3r+2}, x_{3r+3})$$
$$\preccurlyeq \frac{\alpha}{3} Ç(x_{3r}, x_{3r+1}, x_{3r+2}) + \frac{\alpha}{3} Ç(v_{3r}, v_{3r+1}, v_{3r+2}) + \frac{\alpha}{3} Ç(h_{3r}, h_{3r+1}, h_{3r+2})$$

Hence

$$Ç(x_{3r+1}, x_{3r+2}, x_{3r+3}) \preccurlyeq \frac{\frac{\alpha}{3} Ç(x_{3r}, x_{3r+1}, x_{3r+2})}{(1 - P - L - J)} \quad \ldots (8)$$

Also, we can show that

$$Ç(v_{3r+1}, v_{3r+2}, v_{3r+3}) \preccurlyeq \frac{\frac{\alpha}{3} Ç(v_{3r}, v_{3r+1}, v_{3r+2})}{(1 - P - L - J)} \quad \ldots (9)$$

And

$$Ç(h_{3r+1}, h_{3r+2}, h_{3r+3}) \preccurlyeq \frac{\frac{\alpha}{3} Ç(h_{3r}, h_{3r+1}, h_{3r+2})}{(1 - P - L - J)} \quad \ldots (10)$$

Applying (8), (9) and (10) we get

$$Ç(x_{3r+1}, x_{3r+2}, x_{3r+3}) + Ç(v_{3r+1}, v_{3r+2}, v_{3r+3}) + Ç(h_{3r+1}, h_{3r+2}, h_{3r+3})$$
$$\preccurlyeq \frac{\alpha}{(1 - P - L - J)} [Ç(x_{3r}, x_{3r+1}, x_{3r+2}) + Ç(v_{3r}, v_{3r+1}, v_{3r+2}) + Ç(h_{3r}, h_{3r+1}, h_{3r+2})].$$

Therefore

$$Ç(x_{3r+1}, x_{3r+2}, x_{3r+3}) + Ç(v_{3r+1}, v_{3r+2}, v_{3r+3}) + Ç(h_{3r+1}, h_{3r+2}, h_{3r+3})$$
$$\preccurlyeq M[Ç(x_{3r}, x_{3r+1}, x_{3r+2}) + Ç(v_{3r}, v_{3r+1}, v_{3r+2}) + Ç(h_{3r}, h_{3r+1}, h_{3r+2})],$$

where

$$M = \frac{\alpha}{(1 - P - L - J)} < 1$$

and

$$Ç(x_{3r+2}, x_{3r+3}, x_{3r+4}) + Ç(v_{3r+2}, v_{3r+3}, v_{3r+4}) + Ç(h_{3r+2}, h_{3r+3}, h_{3r+4})$$
$$\preccurlyeq M\, Ç(x_{3r+1}, x_{3r+2}, x_{3r+3}) + Ç(v_{3r+1}, v_{3r+2}, v_{3r+3}) + Ç(h_{3r+1}, h_{3r+2}, h_{3r+3})$$
$$\preccurlyeq M^2[Ç(x_{3r}, x_{3r+1}, x_{3r+2}) + Ç(v_{3r}, v_{3r+1}, v_{3r+2}) + Ç(h_{3r}, h_{3r+1}, h_{3r+2})].$$

Continuing with process we have

$$Ç(x_r, x_{r+1}, x_{r+2}) + Ç(v_r, v_{r+1}, v_{r+2}) + Ç(h_r, h_{r+1}, h_{r+2})$$
$$\preccurlyeq M\, Ç(x_{r-1}, x_r, x_{r+1}) + Ç(v_{r-1}, v_r, v_{r+1}) + Ç(h_{r-1}, h_r, h_{r+1})$$
$$\preccurlyeq M^2[Ç(x_{r-2}, x_{r-1}, x_r) + Ç(v_{r-2}, v_{r-1}, v_r) + Ç(h_{r-2}, h_{r-1}, h_r)]$$
$$\preccurlyeq \cdots \preccurlyeq M^r[[Ç(x_0, x_1, x_2) + Ç(v_0, v_1, v_2) + Ç(h_0, h_1, h_2)].$$

If $Ç(x_r, x_{r+1}, x_{r+2}) + Ç(v_r, v_{r+1}, v_{r+2}) + Ç(h_r, h_{r+1}, h_{r+2}) = Ç_r$,
Therefore $Ç_r \preccurlyeq MÇ_{r-1} \preccurlyeq M^2Ç_{r-2} \preccurlyeq \cdots \preccurlyeq M^rÇ_0$.
By definition (2.2) and (9) we get

$$Ç(x_r, x_{r+1}, x_{r+1}) + Ç(v_r, v_{r+1}, v_{r+1}) + Ç(h_r, h_{r+1}, h_{r+1}) \preccurlyeq Ç_r \preccurlyeq M^rÇ_0.$$

For $m > r$,

$$\begin{aligned}
&Ç(x_r, x_m, x_m) + Ç(v_r, v_m, v_m) + Ç(h_r, h_m, h_m) \\
&\preccurlyeq s[Ç(x_r, x_{r+1}, x_{r+1}) + Ç(x_{r+1}, x_m, x_m) + Ç(v_r, v_{r+1}, v_{r+1}) + \\
&Ç(v_{r+1}, v_m, v_m) + Ç(h_r, h_{r+1}, h_{r+1}) + Ç(h_{r+1}, h_m, h_m)] \\
&\preccurlyeq s[Ç(x_r, x_{r+1}, x_{r+1}) + Ç(v_r, v_{r+1}, v_{r+1}) + Ç(h_r, h_{r+1}, h_{r+1})] \\
&+s^2[Ç(x_{r+1}, x_{r+2}, x_{r+2}) + Ç(v_{r+1}, v_{r+2}, v_{r+2}) + Ç(h_{r+1}, h_{r+2}, h_{r+2})] \\
&+ \cdots + s^{m-r}[Ç(x_{m-1}, x_m, x_m) + Ç(v_{m-1}, v_m, v_m) + Ç(h_{m-1}, h_m, h_m)] \\
&\preccurlyeq sM^rÇ_0 + s^2M^{r+1}Ç_0 + \cdots + s^{m-r}M^{m-1}Ç_0 \\
&\prec sM^r[1 + sM + (sM)^2 + \cdots]Ç_0 = \frac{sM^r}{1 - sM}Ç_0 \to 0, \quad as \quad r \to \infty.
\end{aligned}$$

We get that $\{x_r\}$, $\{v_r\}$and $\{h_r\}$ are "Cauchy sequences in X."
By completeness of $Ç_b -$ metric space, so $\exists\, x, v, h \in X$ we get
$x_r \to x, v_r \to v, h_r \to h, \quad as \quad r \to \infty$.
Now let us prove that by contrary
Let $x = S(x, v, h)$, $v = S(v, h, x)$ and $h = S(h, x, v)$,
Suppose that $x \neq S(x, v, h)$, $v \neq (v, h, x)$, $h \neq S(h, x, v)$.
Therefore

$$Ç\big(x, S(x, v, h), S(x, v, h)\big) = l_1 > 0\,, \quad Ç\big(v, S(v, h, x), S(v, h, x)\big) = l_2 > 0\,,$$

and$Ç\big(h, S(h, x, v), S(h, x, v)\big) = l_3 > 0$.
After use inequality (1), we have

$$\begin{aligned}
l_1 &= Ç\big(x, S(x, v, h), S(x, v, h)\big) \\
&\preccurlyeq s[Ç(x, x_{r+1}, x_{r+1}) + Ç\big(x_{r+1}, S(x, v, h), S(x, v, h)\big)] \\
&\preccurlyeq s[Ç(x, x_{r+1}, x_{r+1}) + s[\alpha \frac{Ç(x_r, x, x) + Ç(v_r, v, v) + Ç(h_r, h, h)}{3} \\
&+\beta \frac{Ç(S(x_r, x_r, x_r), S(x, v, h), S(x, v, h))Ç(x_r, x, x)}{1 + Ç(x_r, x, x) + Ç(v_r, v, v) + Ç(h_r, h, h)} \\
&+\gamma \frac{Ç\big(S(x_r, x_r, x_r), S(x, v, h), S(x, v, h)\big), S(v_r, v, v)}{1 + Ç(x_r, x, x) + Ç(v_r, v, v) + Ç(h_r, h, h)} \\
&+\eta \frac{Ç\big(S(x_r, x_r, x_r), S(x, v, h), S(x, v, h)\big)Ç(h_r, h, h)}{1 + Ç(x_r, x, x) + Ç(v_r, v, v) + Ç(h_r, h, h)}].
\end{aligned}$$

Since $\{x_r\}$, $\{v_r\}$and $\{h_r\}$ are convergent to x, v and h,
Then as $r \to \infty$ we take the limit we have
$l_1 \preccurlyeq 0$, which is contradiction, hence $Ç\big(x, S(x, v, h), S(x, v, h)\big) = 0$,
So $x = S(x, v, h)$. Similarly, $v = S(v, h, x)$ and $h = S(h, x, v)$,
And by the same procedures we get
$x = T(x, v, h)$. Similarly, $v = T(v, h, x)$ and $h = T(h, x, v)$
and
$x = R(x, v, h)$. Similarly, $v = R(v, h, x)$ and $h = R(h, x, v)$.
Hence (x, v, h) is "common tripled fixed point" of S, T and R.

Let us prove the existence and this fixed point is unique. assume that
There is (b, l, τ) is the second common fixed point in S, T and R.
Therefore by (7) we get

$$Ç(x,b,b) = Ç\big(S(x,v,h),T(b,l,\tau),R(b,l,\tau)\big)$$
$$\preccurlyeq \alpha \frac{(Ç(x,b,b) + Ç(b,l,l),Ç(h,\tau,\tau))}{3}$$
$$+\ \beta\ \frac{Ç(S(x,v,h),T(b,l,\tau),R(b,l,\tau))Ç(x,b,b)}{1 + Ç(x,b,b) + Ç(w,l,l),Ç(h,\tau,\tau)} + \gamma \frac{Ç\big(S(x,v,h),T(b,l,\tau),R(b,l,\tau)\big)Ç(v,l,l)}{1 + Ç(x,b,b) + Ç(v,l,l),Ç(h,\tau,\tau)}$$
$$+\eta \frac{Ç\big(S(x,v,h),T(b,l,\tau),R(b,l,\tau)\big)Ç(h,\tau,\tau)}{1 + Ç(x,b,b) + Ç(v,l,l),Ç(h,\tau,\tau)}],$$
$$\Longrightarrow$$
$$Ç(x,b,b) \preccurlyeq \frac{\alpha}{3}[Ç(x,b,b) + Ç(v,l,l) + Ç(h,\tau,\tau)] + \betaÇ(x,b,b) + \gammaÇ(x,b.b) + \etaÇ(x,b,b)$$
$$\Rightarrow (1 - \tfrac{\alpha}{3} - \beta - \gamma - \eta)\ Ç(x,b,b) \preccurlyeq \tfrac{\alpha}{3}\ Ç(v,l,l),Ç(h,\tau,\tau),$$
$$\Rightarrow\ Ç(x,b,b) \preccurlyeq \frac{\alpha}{(3-\alpha-3P-\ 3L-3I)}\ Ç(v,l,l),Ç(h,\tau,\tau), \qquad \text{... (11)}$$

Similarly

$$Ç(v,l,l) \preccurlyeq \frac{\alpha}{(3-\alpha-3P-\ 3L-3I)}\ Ç(x,b,b),Ç(h,\tau,\tau), \qquad \text{... (12)}$$

Also $$Ç(h,\tau,\tau) \preccurlyeq \frac{\alpha}{(3-\alpha-3P-\ 3L-3I)}\ Ç(x,b,b),Ç(v,l,l) \qquad \text{... (13)}$$

From (11), (12), (13) we have

$$Ç(x,b,b) + Ç(v,l,l) + Ç(h,\tau,\tau) \preccurlyeq \frac{\alpha}{(3-\alpha-3P-\ 3L-3I)}[Ç(x,b,b) + Ç(v,l,l) + Ç(h,\tau,\tau)]$$

Then $[1 - \frac{\alpha}{(3-\alpha-3P-\ 3L-3I)}]\ [Ç(x,b,b) + Ç(v,l,l) + Ç(h,\tau,\tau)]\ \preccurlyeq 0,$

$$\Longrightarrow \frac{3(1-\alpha-P-\ L-J)}{(3-\alpha-3P-\ 3L-3I)}[Ç(x,b,b) + Ç(v,l,l) + Ç(h,\tau,\tau)]\ \preccurlyeq 0,$$

Since $1 - \alpha - P -\ L - J < 1, \frac{3(1-\alpha-P-\ L-J)}{(3-\alpha-3P-\ 3L-3I)} > 0.$

Therefore

$$Ç(x,b,b) + Ç(v,l,l) + Ç(h,\tau,\tau) = 0$$

That is mean $x = b,\ v = l$ and $h = \tau$ therefore $(x,v,h) = (b,l,\tau)$.
Then S, T *and* R have unique tripled common fixed point.■

Theorem 4.5. Let X =C $([\dot{\sigma}, \tilde{\omega}], R^r)$, $\dot{\sigma} > 0$ and $Ç_b: X \times X \times X \to C$ is a mapping satisfying:
$Ç_b(x,v,h) =\ max_{\zeta\in[\dot{\sigma},\tilde{\omega}]}\big(\left||x(\zeta) - v(\zeta)\right|| + \left||v(\zeta) - h(\zeta)\right|| + \left||x(\zeta) - h(\zeta)\right||\big)\sqrt{1+\dot{\sigma}^2}e^{itan^{-1}\dot{\sigma}}$,
for all $x, v, h \in X$, and $\zeta \in [\dot{\sigma}, \tilde{\omega}] \subset R$.

According to Urysohn's integral equations in (12)
Assume that U_1, U_2 and U_3:$[\dot{\sigma}, \tilde{\omega}] \times [\dot{\sigma}, \tilde{\omega}] \times [\dot{\sigma}, \tilde{\omega}] \to R^r$ such that F_x, G_x and $H_x \in X$, for each $x \in X$.

Where

$$F_x(\zeta) = \int_{\dot{\sigma}}^{\tilde{\omega}} U_1\big(\zeta,s,x(\zeta)\big)ds,\ \ G_x(\zeta) = \int_{\dot{\sigma}}^{\tilde{\omega}} U_2\big(\zeta,s,x(\zeta)\big)ds,$$
$$H_x(\zeta) = U_3\big(\zeta,s,x(\zeta)\big)$$

if $\exists\ s \geq 1$, $\lambda \in (0,1)$ consider the inequality

$$A(x,v,h)(\zeta) \precsim \frac{\lambda}{s^3} R(x,v,h)(\zeta), \qquad \text{... (14)}$$

Where

$$R(x,v,h) = max\{D(x,v,h)(\zeta),B(x,v,h)(\zeta),C(x,v,h)(\zeta),E(x,v,h)(\zeta),$$

$$\frac{1}{3}[B(x,v,h)(\zeta)+C(x,v,h)(\zeta)+E(x,v,h)(\zeta)],$$
$$\frac{B(x,v,h)(\zeta),C(x,v,h)(\zeta),E(x,v,h)(\zeta)}{1+D(x,v,h)(\zeta)}\},$$

and

$$A(x,v,h)(\zeta)=\|(F_x(\zeta)-G_v(\zeta))+(G_v(\zeta)-H_h(\zeta))\ +(f(\zeta)-k(\zeta))+(k(\zeta)-p(\zeta))\ \|\sqrt{1+\dot{\sigma}^2}e^{itan^{-1}\dot{\sigma}},$$
$$B(x,v,h)(\zeta)=||x(\zeta)-F_x(\zeta))-f(\zeta))||\sqrt{1+\dot{\sigma}^2}e^{itan^{-1}\dot{\sigma}},$$
$$C(x,v,h)(\zeta)=\|v(\zeta)-G_v(\zeta)-k(\zeta)\|\sqrt{1+\dot{\sigma}^2}e^{itan^{-1}\dot{\sigma}},$$
$$E(x,v,h)(\zeta)=\|\ h(\zeta)-H_h(\zeta)-p(\zeta)\|\sqrt{1+\dot{\sigma}^2}e^{itan^{-1}\dot{\sigma}},$$
$$D(x,v,h)(\zeta)=\|\ \big(x(\zeta)-v(\zeta)\big)+(v(\zeta)-h(\zeta))\|\sqrt{1+\dot{\sigma}^2\zeta}e^{itan^{-1}\dot{\sigma}},$$

holds for all $x,v,h\in X$. therefore, “system of Urysohn’s integral equations” has unique common solution in X .

Proof: First define two mappings $S,T,W:X\ \to X$ by $S_x=F_x+f,\ T_x=G_x+k, W_x=H_x+p$. Then,

$$Ç_b(S_x,T_x,W_x)=$$
$$\|(F_x(\zeta)-G_v(\zeta))+(G_v(\zeta)-H_h(\zeta))\ +(f(\zeta)-k(\zeta))+(k(\zeta)-p(\zeta))\ \|_\infty\ \sqrt{1+\dot{\sigma}^2}e^{itan^{-1}\dot{\sigma}},$$
$$Ç_b(x,S_x,S_x)=\|x(\zeta)-F_x(\zeta)-f(\zeta)\|_\infty\ \sqrt{1+\dot{\sigma}^2}e^{itan^{-1}\dot{\sigma}},$$

$$Ç_b(v,S_v,S_v)=\|v(\zeta)-G_v(\zeta)-h(\zeta)\|_\infty\ \sqrt{1+\dot{\sigma}^2}e^{itan^{-1}\dot{\sigma}},$$
$$Ç_b(h,T_h,T_h)=max_{\zeta\in[\dot{\sigma},\tilde{\omega}]}\|\ h(\zeta)-H_h(\zeta)-p(\zeta)\|_\infty\sqrt{1+\dot{\sigma}^2}e^{itan^{-1}\dot{\sigma}},$$
$$Ç_b(x,v,h)=max_{\zeta\in[\dot{\sigma},\tilde{\omega}]}\|\big(x(\zeta)-v(\zeta)\big)+\big(v(\zeta)-h(\zeta)\big)\|_\infty\sqrt{1+\dot{\sigma}^2}e^{itan^{-1}\dot{\sigma}}.$$

From assumption (9), for each $\zeta\in[\dot{\sigma},\tilde{\omega}]$ we have:

$$A(x,v,h)(\zeta)\precsim\frac{\lambda}{s^3}R(x,v,h)(\zeta)$$
$$\precsim\frac{\lambda}{s^3}max\{D(x,v,h)(\zeta),B(x,v,h)(\zeta),C(x,v,h)(\zeta),E(x,v,h)(\zeta),$$
$$\frac{1}{3}[B(x,v,h)(\zeta)+C(x,v,h)(\zeta)+E(x,v,h)(\zeta)],$$
$$\frac{B(x,v,h)(\zeta),C(x,v,h)(\zeta),E(x,v,h)(\zeta)}{1+D(x,v,h)(\zeta)}\},$$

Which implies that

$$max_{\zeta\in[\dot{\sigma},\tilde{\omega}]}A(x,v,h)(\zeta)$$
$$\precsim\frac{\lambda}{s^3}max_{\zeta\in[\dot{\sigma},\tilde{\omega}]}\ max\{D(x,v,h)(\zeta),B(x,v,h)(\zeta),C(x,v,h)(\zeta),E(x,v,h)(\zeta),$$
$$\frac{1}{3}[B(x,v,h)(\zeta)+C(x,v,h)(\zeta)+E(x,v,h)(\zeta)],$$

$$\frac{B(x,v,h)(\zeta),C(x,v,h)(\zeta),E(x,v,h)(\zeta)}{1+D(x,v,h)(\zeta)}\},$$
$$\precsim\frac{\lambda}{s^3}max\{max_{\zeta\in[\dot{\sigma},\tilde{\omega}]}D(x,v,h)(\zeta),max_{\zeta\in[\dot{\sigma},\tilde{\omega}]}B(x,v,h)(\zeta),$$
$$max_{\zeta\in[\dot{\sigma},\tilde{\omega}]}C(x,v,h)(\zeta),max_{\zeta\in[\dot{\sigma},\tilde{\omega}]}E(x,v,h)(\zeta),$$

$$\frac{1}{3}\left[max_{\zeta\in[\dot{\sigma},\tilde{\omega}]}B(x,v,h)(\zeta)+max_{\zeta\in[\dot{\sigma},\tilde{\omega}]}C(x,v,h)(\zeta)+max_{\zeta\in[\dot{\sigma},\tilde{\omega}]}E(x,v,h)(\zeta)\right],$$
$$\frac{max_{\zeta\in[\dot{\sigma},\tilde{\omega}]}B(x,v,h)(\zeta), max_{\zeta\in[\dot{\sigma},\tilde{\omega}]}C(x,v,h)(\zeta), max_{\zeta\in[\dot{\sigma},\tilde{\omega}]}E(x,v,h)(\zeta)}{1+max_{\zeta\in[a,b]}D(x,v,h)(\zeta)}\},$$

Therefore,

$$Ç_b(S_x,T_v,W_h)\precsim\frac{\lambda}{s^3}max\{Ç_b(x,v,h),Ç_b(x,S_x,S_x),Ç_b(v,T_v,T_v),Ç_b(h,W_h,W_h),$$
$$\frac{1}{3}[Ç_b(x,T_v,W_h)(\zeta)+Ç_b(w,S_x,W_h)(\zeta)+Ç_b(h,T_v,S_x)(\zeta)],$$
$$\frac{Ç_b(x,S_x,S_x),d(v,T_v,T_v),d(h,W_h,W_h)}{1+Ç_b(x,v,h)}\}.$$

Hence, every condition (Theorem 4.4) with $\psi=R=\zeta=I_x$.

From the foregoing, we were able to prove the unity of the system of Urysohn which pertains integral equations has unique common solution in X. ■

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المستطيلة ذات القيم المعقدة: $Ç_b$ الفضاءات المترية وخصائصها طوبولوجيا وتطبيقاتها

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معلومات المؤلف

الايميل:

الموبايل :

الخلاصة:

في هذا البحث نقدم اولا مفهوم الفضاء المتري $Ç_b$ المستطيل ذو القيمة المعقد والذي يعمم مفهوم الفضاء المتري المستطيل ذو القيمة المعقدة. بعد ذلك، ت الحصول على بعض الخصائص المترية والطوبولوجيا، ونتائج النقطة الصامد والفضاء –b المتري المستطيل ذو القيمة المعقدة والأمثلة المرتبطة ببعض التطبيقات الانكماشية في حالة مساحات مترية $Ç_b$ مستطيلة ذات قيمة معقدة بالإضافة إلى ذلك، سوف نقوم بتعزيز النتيجة الرئيسية من خلال التطبيق، حيث يتم إعطاء الحل المشترك الوحيد لنظام معادلة أوريسون التكاملية.