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On Nearly Semi-2-Absorbing Submodules

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Abstract

This research argues that a commutative ring with an identity and D is a unitary left R-module. Nearly Semi-2-Absorbing sub-modules as a new generalization of 2-Absorbing and Semi-2-Absorbing sub-modules were introduced in this paper. Many basic properties, characterizations, and examples of this concept are given. Moreover, several characterizations of Nearly Semi-2-Absorbing sub-modules in the class of multiplication modules have been established. Furthermore, we show that by examples, the residual of Nearly Semi-2-Absorbing sub-modules does not need to be Nearly Semi-2-Absorbing ideals of R. So, we have proven this case in some types of modules.

Introduction:

A well-known concept to start with is 2-Absorbing sub-modules, where a proper sub-module S of an R-module D is called 2-Absorbing if whenever $abd \in S$, for $a, b \in$ R, $d \in D$, implies either $ad \in S$ or $bd \in S$ or $abD \subseteq S$ [1]. This concept was generalized to Semi-2-Absorbing, Primary-2-Absorbing, and Quasi-2-Absorbing sub-modules see [2, 3, and 4]. Recently the concept of 2-Absorbing sub-modules was generalized to (Pseudo-2-Absorbing, Pseudo Semi-2-Absorbing, Pseudo Quasi-2-Absorbing, and Pseudo Primary-2-Absorbing) sub-modules see [5, 6, and 7]. In this research, we introduce a new generalization of 2-Absorbing sub-modules, which we called Nearly Semi-2-Absorbing sub-modules, and we show by example every 2-Absorbing (Semi-2-Absorbing) sub-module is Nearly Semi-2-Absorbing sub-module, but not conversely.

Basic Concept:

In this part, we recall some basic definitions, proportions, and remarks that we need themes in the sequel.

Definition (2.1)[2]. A proper sub-module S of an Ȑ-module D is called Semi-2-Absorbing if whenever $a^2d \in S$, for $a \in$ Ȑ, $d \in D$, implies either $ad \in S$ or $a^2D \subseteq S$.

Definition (2.2)[8]. The Jacobson radical of an Ȑ-module D is denoted by $\mathcal{J}(D)$, defined as the intersection of all maximal sub-modules of D.

Definition (2.3)[10]. An Ȑ-module D is called semi-simple if each sub-module of D is a direct summand of D.

Proposition (2.4)[8, Ex. (12)]. If S is a sub-module of an Ȑ-module$D$, such that S is a direct summand of D, then $\mathcal{J}\left(\frac{D}{S}\right) = \frac{\mathcal{J}(D)+S}{S}$.

Definition (2.5)[11]. $[S:_D I] = \{d \in D: dI \subseteq S\}$, where S is a sub-module of an Ȑ-module D, and I is a non-zero ideal of Ȑ. $[S:_D I]$ is a sub-module of D containing S and $[I:\text{Ȑ}] = I$.

Definition (2.6)[12]. An Ȑ-module D is multiplication if each sub-module S of D is of the form $S = ID$ for some ideal I of Ȑ. Equivalently, D is multiplication if $S = [S:_{\text{Ȑ}} D]D$.

Definition (2.7)[2]. A sub-module S and L of a multiplication Ȑ-module D with $S = ID$, $L = \mathcal{J}D$, for some ideals I and $\mathcal{J}$ of Ȑ. The product $SL = IS.\mathcal{J}L = I\mathcal{J}D$, that is $SL = IL$, in particular $SD = IDD = ID = S$. Also for any $d \in D$ we have $d = \text{Ȑ}d$ as a sub-module of D.

Definition (2.8)[8]. An Ȑ-module D is a projective if for any Ȑ-epimorphism f from an Ȑ-module D on to an Ȑ-module $\overline{D}$ and for any homomorphism g from an Ȑ-module $\overline{\overline{D}}$ to $\overline{D}$, there exists a homomorphism h from $\overline{\overline{D}}$ to D such that $f \circ h = g$.

Proposition (2.9) [8, The. (9.2.1)(g)]. If D is a projective Ȑ-module, then $\mathcal{J}(\text{Ȑ})D = \mathcal{J}(D)$.

Definition (2.10)[8]. An Ȑ-module D is faithful if $ann(D) = \{r \in \text{Ȑ}: rd = (0)\} = (0)$.

Proposition (2.11)[13, Rem. p.14]. Let D be faithful multiplication Ȑ-module, then $\mathcal{J}(D) = \mathcal{J}(\text{Ȑ})D$.

Definition (2.12)[14]. An Ȑ-module D is said content if $(\bigcap_{i\in I} A_i)D = \bigcap_{i\in I} A_i D$, for some family of ideals A_i in Ȑ.

Proposition (2.13)[13, pro. (1.11)]. If D is content Ȑ-module, then $\mathcal{J}(D) = \mathcal{J}(\text{Ȑ})D$.

Definition (2.14)[8]. Ȑ is a good ring if $\mathcal{J}(\text{Ȑ})D = \mathcal{J}(D)$.

Definition (2.15)[11]. A ring Ȑ is said to be local ring if Ȑ has a unique maximal ideal.

Proposition (2.16) [15, pro. (1.12)]. If D is an Ȑ-module over local ring Ȑ, then $\mathcal{J}(\text{Ȑ})D = \mathcal{J}(D)$.

Proposition (2.17) [8, cor. (9.7.3)(b)]. If R is an Artinian ring, then R is a good ring.

Proposition (2.18)[16, cor. (9)]. If D is a finitely-generated multiplication R-module, I_1 and I_2 are ideals in R. Then $I_1 D \subseteq I_2 D$ if and only if $I_1 \subseteq I_2 + ann_{\mathrm{R}}(D)$.

Definition (2.19)[17]. A ring R is v-ring if for any R-module $\mathcal{J}(D) = 0$.

Definition (2.20)[18]. An R-module D is called regular if $\frac{\mathrm{R}}{ann(d)}$ is regular $\forall d \in D$.

Proposition (2.21)[19, pro. (3.9)]. Let D be a regular module, then $\mathcal{J}(D) = 0$.

Proposition (2.22)[8, pro. (9.1.4)(b)]. If S is a sub-module of an R-module D, with $\mathcal{J}\left(\frac{D}{S}\right) = 0$, then $\mathcal{J}(D) \subseteq S$.

Nearly Semi-2-Absorbing Sub-modules:

This section introduces the definition of the Nearly Semi-2-Absorbing submodule, which appears in [9], and gives several basic properties, characterizations, and examples of this notion.

Definition (3.1): A proper sub-module S of an R-module D is said to be Nearly Semi-2-Absorbing sub-module of D, if whenever $r^2 d \in S$, for $r \in \mathrm{R}$, $d \in D$, implies either $rd \in S + \mathcal{J}(D)$ or $r^2 d \in [S + \mathcal{J}(D){:}\, D]$. And a proper ideal $\mathcal{J}$ of a ring R is said to be Nearly Semi-2-Absorbing ideal if $\mathcal{J}$ is Nearly Semi-2-Absorbing sub-module of an R-module R.

Remark (3.2): It's obvious that every 2-Absorbing sub-module is Nearly Semi-2-Absorbing, but contrariwise isn't true as in the example.

Example (3.3): Consider the Z-module Z_{48} and sub-module $S = \langle\bar{8}\rangle$. It is apparent that S is Nearly a Semi-2-Absorbing sub-module because $S + \mathcal{J}(Z_{48}) = \langle\bar{8}\rangle + \langle\bar{6}\rangle = \langle\bar{2}\rangle$, but S is not a 2-Absorbing sub-module of Z_{48}. Since $2.2.\bar{2} \in S$, for $2 \in Z$, $\bar{2} \in Z_{48}$, implies $2.\bar{2} \notin S$ and $2.2.(Z_8) \nsubseteq S$.

Remark (3.4): It's obvious that every Semi-2-Absorbing sub-module is Nearly Semi-2-Absorbing, but contrariwise isn't true as in the example.

Example (3.5): The sub-module $S = \langle\overline{12}\rangle$ is Nearly Semi-2-Absorbing sub-module of Z_{48}, because $S + \mathcal{J}(Z_{48}) = \langle\overline{12}\rangle + \langle\bar{6}\rangle = \langle\bar{2}\rangle$ but S is not Semi-2-Absorbing, because $2^2.\bar{3} \in S$, for $2 \in Z$, $\bar{3} \in Z_{48}$, implies $2.\bar{3} \notin S$ and $2^2(Z_8) \nsubseteq S$.

The propositions that follow are characterizations of Nearly Semi-2-Absorbing sub-modules.

Proposition (3.6): A proper sub-module S of an R-module D is Nearly Semi-2-Absorbing sub-module of D if and only if for any $r \in \mathrm{R}$ such that $r^2 \notin [S + \mathcal{J}(D) :_{\mathrm{R}} D]$, then $[S :_D r^2] \subseteq [S + \mathcal{J}(D) :_D r]$.

Proof: ⇒) let $d \in [S :_D r^2]$, then $r^2 d \in S$. Since S is Nearly Semi-2-Absorbing sub-module of D and $r^2 \notin [S + \mathcal{J}(D) :_R D]$, it follows that $rd \in S + \mathcal{J}(D)$. Thus $d \in [S + \mathcal{J}(D) :_D r]$. Therefore $[S :_D r^2] \subseteq [S + \mathcal{J}(D) :_D r]$.

⇐) Let $r^2 d \in S$ for $r \in R$, $d \in D$ and let $r^2 \notin [S + \mathcal{J}(D) :_R D]$. But $d \in [S :_D r^2] \subseteq [S + \mathcal{J}(D) :_D r]$. It follows that $d \in [S + \mathcal{J}(D) :_D r]$, that is $rd \in S + \mathcal{J}(D)$. Hence S Nearly Semi-2-Absorbing sub-module of D.

Proposition (3.7): Let S be a proper sub-module of an R-module$D$. Then S is a Nearly Semi-2-Absorbing sub-module of D if and only if $I^2 L \subseteq S$ for I is an ideal of R and L is a sub-module of D, which implies that either $IL \subseteq S + \mathcal{J}(D)$ or $I^2 \subseteq [S + \mathcal{J}(D) :_R D]$.

Proof: ⇒) Let $I^2 L \subseteq S$ for I is an ideal of R and L is a sub-module of D, with $I^2 \nsubseteq [S + \mathcal{J}(D) :_R D]$ and S is Nearly Semi-2-Absorbing sub-module of D. To prove that $IL \subseteq S + \mathcal{J}(D)$. Let $y \in IL$, implies that $y = r_1 y_1 + r_2 y_2 + \cdots + r_n y_n$ for $r_j \in I$ and $y_j \in L$, $j = 1,2,\dots,n$, it follows ${r_j}^2 y_j \in I^2 L \subseteq S$. That is ${r_j}^2 y_j \in S$. But S is Nearly Semi-2-Absorbing sub-module of D, then $r_j y_j \in S + \mathcal{J}(D)$ and ${r_j}^2 \notin [S + \mathcal{J}(D) :_R D]$ for each $j = 1,2,3,\dots,n$, thus $r_1 y_1 + r_2 y_2 + \cdots + r_n y_n \in S + \mathcal{J}(D)$, that is $y \in S + \mathcal{J}(D)$. Hence $IL \subseteq S + \mathcal{J}(D)$.

⇐) Suppose that $r^2 d \in S$ for $r \in R$ and $d \in D$, implies that $\langle r^2 \rangle \langle d \rangle \subseteq S$. Thus by our assumption we have either $\langle r \rangle \langle d \rangle \subseteq S + \mathcal{J}(D)$ or $\langle r^2 \rangle \subseteq [S + \mathcal{J}(D) :_R D]$. That is $rd \in \langle r \rangle \langle d \rangle \subseteq S + \mathcal{J}(D)$ or $r^2 \in \langle r^2 \rangle \subseteq [S + \mathcal{J}(D) :_R D]$, that is either $rd \in S + \mathcal{J}(D)$ or $r^2 \in [S + \mathcal{J}(D) :_R D]$. Hence S is Nearly Semi-2-Absorbing sub-module of$D$.

The following corollaries result from a direct application of proposition (3.7).

Corollaries (3.8): Let S be a proper sub-module of an R-module D. Then S is Nearly Semi-2-Absorbing sub-module of D if and only if $r^2 L \subseteq S$ for $r \in R$ and L is a sub-module of D, implies that either $rL \subseteq S + \mathcal{J}(D)$ or $r^2 \subseteq [S + \mathcal{J}(D) :_R D]$.

Corollary (3.9): Let S be the proper sub-module of an R-module$D$. Then S is Nearly a Semi-2-Absorbing sub-module of D if and only if $I^2 d \subseteq S$ for I is an ideal of R and $d \in D$, implies that either $Id \subseteq S + \mathcal{J}(D)$ or $I^2 \subseteq [S + \mathcal{J}(D) :_R D]$.

The following results are some basic properties of Nearly Semi-2-Absorbing sub-modules.

Proposition (3.10): Let S be the Nearly Semi-2-Absorbing sub-module of an R-module D, and L is a sub-module of D with $L \subseteq S$, then $\frac{S}{L}$ is Nearly Semi-2-Absorbing sub-module of an R-module $\frac{D}{L}$.

Proof: Let $r^2(d + L) = r^2 d + L \in \frac{S}{L}$ for $r \in R$, $d + L \in \frac{D}{L}$, and $d \in D$, implies that $r^2 d \in S$. Since S is Nearly a Semi-2-Absorbing sub-module of R, then either $rd \in S + \mathcal{J}(D)$ or $r^2 D \subseteq S + \mathcal{J}(D)$. It follows that, either $r(d + L) \in \frac{S+\mathcal{J}(D)}{L}$ or $r^2 \frac{D}{L} \subseteq \frac{S+\mathcal{J}(D)}{L}$, that is either $r(d + L) \in \frac{S}{L} + \frac{S+\mathcal{J}(D)}{L} \subseteq \frac{S}{L} + \mathcal{J}(\frac{D}{L})$ or $r^2 \frac{D}{L} \subseteq \frac{S}{L} + \frac{S+\mathcal{J}(D)}{L} \subseteq \frac{S}{L} + \mathcal{J}(\frac{D}{L})$. Hence, $\frac{S}{L}$ is Nearly a Semi-2-Absorbing sub-module of $\frac{D}{L}$.

Proposition (3.11): Let S and K be sub-modules for a semisimple R-module D, such that $K \subseteq S$ and S is a proper sub-module of D. If K and $\frac{S}{K}$ are Nearly Semi-2-Absorbing sub-modules of D and $\frac{D}{K}$ respectively, then S is Nearly Semi-2-Absorbing sub-module of$D$.

Proof: Suppose K and $\frac{S}{K}$ are Nearly Semi-2-Absorbing sub-modules for D and $\frac{D}{K}$, respectively, and let $I^2d \subseteq S$, for I is an ideal of R and $d \in D$. So $I^2(d+K) = I^2d + K \subseteq \frac{S}{K}$. If $I^2d \subseteq K$ and K is Nearly Semi-2-Absorbing sub-modules of D, implies that by corollary (3.9) either $Id \subseteq K + \mathcal{J}(D) \subseteq S + \mathcal{J}(D)$ or $I^2D \subseteq K + \mathcal{J}(D) \subseteq S + \mathcal{J}(D)$, hence S is Nearly Semi-2-Absorbing sub-modules for D. Now, we may assume that $I^2d \nsubseteq K$. It follows that $I^2(d+K) \subseteq \frac{S}{K}$, but $\frac{S}{K}$ is Nearly Semi-2-Absorbing sub-modules of $\frac{D}{K}$ again by corollary (3.9) either $I(d+K) \subseteq \frac{S}{K} + \mathcal{J}\left(\frac{D}{K}\right)$ or $I^2\frac{D}{K} \subseteq \frac{S}{K} + \mathcal{J}\left(\frac{D}{K}\right)$. Since D is semi-simple, that is, every sub-module is a direct summand, and hence by proposition (2.4), either $I(d+K) \subseteq \frac{S}{K} + \frac{K+\mathcal{J}(D)}{K}$ or $I^2\frac{D}{K} \subseteq \frac{S}{K} + \frac{K+\mathcal{J}(D)}{K}$. But $K \subseteq S$, it follows that $K + \mathcal{J}(D) \subseteq S + \mathcal{J}(D)$, hence $\frac{S}{K} + \frac{K+\mathcal{J}(D)}{K} \subseteq \frac{S}{K} + \frac{S+\mathcal{J}(D)}{K} = \frac{S+\mathcal{J}(D)}{K}$. Thus either $I(d+K) \subseteq \frac{S+\mathcal{J}(D)}{K}$ or $I^2\frac{D}{K} \subseteq \frac{S+\mathcal{J}(D)}{K}$, it follows that either $Id \subseteq S + \mathcal{J}(D)$ or $I^2D \subseteq S + \mathcal{J}(D)$. Hence, by corollary (3.9), S is Nearly Semi-2-Absorbing sub-modules of D.

Proposition (3.12): Let D be an R-module with $\mathcal{J}(D)$ as a Semi-2-Absorbing sub-module of D. If S is a proper sub-module of D such that $S \subseteq \mathcal{J}(D)$, then S is Nearly Semi-2-Absorbing sub-module of D.

Proof: Let $r^2L \subseteq S$ for $r \in$ R and L is a sub-module of D. Since $S \subseteq \mathcal{J}(D)$, so $r^2L \subseteq \mathcal{J}(D)$. However, $\mathcal{J}(D)$ is Semi-2-Absorbing sub-module, then either $rL \subseteq \mathcal{J}(D) \subseteq S + \mathcal{J}(D)$ or $r^2D \subseteq \mathcal{J}(D) \subseteq S + \mathcal{J}(D)$. That is either $rL \subseteq S + \mathcal{J}(D)$ or $r^2 \in [S + \mathcal{J}(D):_R D]$. Hence by Corollary (3.8) S is Nearly Semi-2-Absorbing sub-module of D.

Proposition (3.13): Let $D = D_1 \oplus D_2$ be an R-module where D_1, D_2 are R-modules and $S = S_1 \oplus S_2$be a sub-module of D, where S_1, S_2 are sub-modules of D_1, D_2 respectively and $S \subseteq \mathcal{J}(D)$. If S is Nearly Semi-2-Absorbing sub-module of D, then S_1, S_2 are Nearly Semi-2-Absorbing sub-modules of D_1, D_2 respectively.

Proof: let $r^2d_1 \in S_1$ for $r \in$ R, $d_1 \in D_1$. It follows that $r^2(d_1, 0) \in S_1 \oplus S_2$. But $S = S_1 \oplus S_2$ is Nearly Semi-2-Absorbing sub-module, then either $r(d_1, 0) \in S_1 \oplus S_2 + \mathcal{J}(D_1 \oplus D_2)$ or $r^2D \subseteq S_1 \oplus S_2 + \mathcal{J}(D_1 \oplus D_2)$. But $S \subseteq \mathcal{J}(D)$, then $S + \mathcal{J}(D) = \mathcal{J}(D) = \mathcal{J}(D_1 \oplus D_2) = \mathcal{J}(D_1) \oplus \mathcal{J}(D_2)$. Thus either $r(d_1, 0) \in \mathcal{J}(D_1) \oplus \mathcal{J}(D_2)$ or $r^2D \subseteq \mathcal{J}(D_1) \oplus \mathcal{J}(D_2)$. It follows that either $rd_1 \in \mathcal{J}(D_1) \subseteq S_1 + \mathcal{J}(D_1)$ or $r^2D \subseteq \mathcal{J}(D_1) \subseteq S_1 + \mathcal{J}(D_1)$. Hence S_1 is Nearly Semi-2-Absorbing sub-modules of D_1.

In the same way S_2 is Nearly Semi-2-Absorbing sub-modules of D_2.

Proposition (3.14): Let $D = D_1 \oplus D_2$ be an R-module with D_1, D_2 are R-modules and S be a sub-module of D_1, with $\mathcal{J}(D_1) \subseteq S_1$ and $\mathcal{J}(D_1 \oplus D_2) \subseteq S_1 \oplus D_2$. Then S_1 is a Nearly Semi-2-Absorbing sub-module of D_1 if and only if $S_1 \oplus D_2$ is a Nearly Semi-2-Absorbing sub-module of D.

Proof: ⇒) Suppose that S_1 is Nearly Semi-2-Absorbing sub-module of D_1 with $\mathcal{J}(D_1) \subseteq S_1$ and $\mathcal{J}(D_1 \oplus D_2) \subseteq S_1 \oplus D_2$ and let $r^2(d_1, d_2) \in S_1 \oplus D_2$ for $r \in$ R, $(d_1, d_2) \in D_1 \oplus D_2$, where $d_1 \in D_1$ and $d_2 \in D_2$, implies that $r^2 d_1 \in S_1$ and $r^2 d_2 \in D_2$. by hypothesis either $rd_1 \in S + \mathcal{J}(D_1)$ or $r^2 D_1 \subseteq S + \mathcal{J}(D_1)$.But $\mathcal{J}(D_1) \subseteq S_1$, then $S_1 + \mathcal{J}(D_1) = S_1$, it follows that either $rd_1 \in S_1$ or $r^2 D_1 \subseteq S_1$, thus either $r(d_1, d_2) \in S_1 \oplus D_2 = S_1 \oplus D_2 + \mathcal{J}(D_1 \oplus D_2)$ or $r^2 D \subseteq S_1 \oplus D_2 = S_1 \oplus D_2 + \mathcal{J}(D_1 \oplus D_2)$. Hence $S_1 \oplus D_2$ is Nearly Semi-2-Absorbing sub-modules of $D_1 \oplus D_2$.

⇐) Now, assume $S_1 \oplus D_2$ is Nearly Semi-2-Absorbing sub-modules of $D_1 \oplus D_2$, with $\mathcal{J}(D_1 \oplus D_2) \subseteq S_1 \oplus D_2$ and $r^2 d_1 \in S_1$ for $r \in$ R, $d_1 \in D_1$. implies that $r^2(d_1, d_2) \in S_1 \oplus D_2$ for each $d_2 \in D_2$. By hypothesis either $r(d_1, d_2) \in S_1 \oplus D_2 + \mathcal{J}(D_1 \oplus D_2) = S_1 \oplus D_2$ or $r^2(D_1 \oplus D_2) \subseteq S_1 \oplus D_2 + \mathcal{J}(D_1 \oplus D_2) = S_1 \oplus D_2$. Then either $rd_1 \in S_1 \subseteq S_1 + \mathcal{J}(D_1)$ or $r^2 D_1 \subseteq S_1 \subseteq S_1 + \mathcal{J}(D_1)$. Hence S_1 is a Nearly Semi-2-Absorbing sub-module of D_1.

Proposition (3.15): Let $D = D_1 \oplus D_2$ be an R-module with D_1, D_2 be an R-modules and S be a sub-module of D_1, with $\mathcal{J}(D_2) \subseteq S_2$ and $\mathcal{J}(D_1 \oplus D_2) \subseteq D_1 \oplus S_2$. Then S_2 is Nearly Semi-2-Absorbing sub-module of D_2 if and only if $D_1 \oplus S_2$ is Nearly Semi-2-Absorbing sub-modules of D.

Proof: Similarly as in proposition (3.14).

Proposition (3.16): Let D be an R-module, and S be a proper sub-module of D, such that $\mathcal{J}(D) \subseteq S$. Then S is Nearly a Semi-2-Absorbing sub-module of D if and only if $[S :_D I]$ is Nearly a Semi-2-Absorbing sub-module of D for each ideal I of R.

Proof: ⇒) Let $r^2 K \subseteq [S :_D I]$ for $r \in$ R, K is a sub-module of D, hence $r^2(IK) \subseteq S$. But S is Nearly Semi-2-Absorbing sub-module of D, implies either $r(IK) \subseteq S + \mathcal{J}(D)$ or $r^2 D \subseteq S + \mathcal{J}(D)$. But $\mathcal{J}(D) \subseteq S$, then $S + \mathcal{J}(D) = S$. Thus $rIK \subseteq S$, then either $rK \subseteq [S :_D I]$ or $r^2 D \subseteq S \subseteq [S :_D I]$. That is either $rK \subseteq [S :_D I] + \mathcal{J}(D)$ or $r^2 D \subseteq [S :_D I] + \mathcal{J}(D)$. Hence $[S :_D I]$ is Nearly Semi-2-Absorbing sub-module of D.

⇐) Suppose $[S :_D I]$ is Nearly Semi-2-Absorbing sub-module of D, for every non-zero ideal I of R. Put $I =$ R, we get $[S :_D \text{R}] = S$ is Nearly Semi-2-Absorbing sub-module of D.

Characterizations of Nearly Semi-2-Absorbing Sub-modules in a class of Multiplication Modules:

In this part of the research, we introduce different characterizations of the Nearly Semi-2-Absorbing sub-module in a class of multiplication modules.

Proposition (4.1): A proper sub-module S of a multiplication R-module D is Nearly Semi-2-Absorbing if and only if $H^2 K \subseteq S$ for H and K are sub-modules of D, which implies that either $HK \subseteq S + \mathcal{J}(D)$ or $H^2 \subseteq S + \mathcal{J}(D)$.

Proof: ⇒) Let $H^2 K \subseteq S$ for H, K are sub-modules of multiplication module D; it follows $(ID)^2(\mathcal{I}D) = I^2 \mathcal{I} D \subseteq S$ for some ideals I, $\mathcal{I}$ in R. Since S is Nearly Semi-2-Absorbing sub-module of D, then by proposition (3.7) we have either $I\mathcal{I}D \subseteq S + \mathcal{J}(D)$ or $I^2 \subseteq [S + \mathcal{J}(D) :_{\text{R}} D]$, that is either $HK \subseteq S + \mathcal{J}(D)$ or $H^2 \subseteq S + \mathcal{J}(D)$.

⇐) Let $r^2K \subseteq S$ for $r \in$ R, K is a sub-module of D. But D is a multiplication module, so $K= ID$ for some ideal I of R, it follows that $r^2ID \subseteq S$, hence by hypothesis either $rID \subseteq S + \mathcal{J}(D)$ or $r^2 \in [S + \mathcal{J}(D) :_{\text{R}} D]$. That is either $rK \subseteq S + \mathcal{J}(D)$ or $r^2 \in [S + \mathcal{J}(D) :_{\text{R}} D]$. Hence S is Nearly Semi-2-Absorbing sub-module of D.

Proposition (4.2): A proper sub-module S of a multiplication R-module D is Nearly Semi-2-Absorbing if and only if ${d_1}^2 d_2 \subseteq S$ for d_1, $d_2 \in D$, implies that either $d_1 d_2 \subseteq S + \mathcal{J}(D)$ or ${d_1}^2 \subseteq S + \mathcal{J}(D)$.

Proof: ⇒) Let ${d_1}^2 d_2 \subseteq S$ for $d_1, d_2 \in D$, it follows that $(d_1)^2(d_2) \subseteq S$. But D is a multiplication module, then $(d_1)^2= (ID)^2 = I^2D$ and $(d_2) = \mathcal{J}D$ for some ideals I and $\mathcal{J}$ in R, then $I^2\mathcal{J}D \subseteq S$, since S is Nearly Semi-2-Absorbing sub-module, then by proposition (3.7) either $I\mathcal{J}D \subseteq S + \mathcal{J}(D)$ or $I^2D \subseteq S + \mathcal{J}(D)$. That is either $d_1 d_2 \subseteq S + \mathcal{J}(D)$ or ${d_1}^2 \subseteq S + \mathcal{J}(D)$.

⇐) Clear.

The following corollaries result from a direct application of proposition (4.1).

Corollary (4.3): A proper sub-module S of a multiplication R-module D is Nearly Semi-2-Absorbing if and only if $H^2k \subseteq S$ for H is a sub-module of D and $k \in D$, implies that either $Hk \subseteq S + \mathcal{J}(D)$ or $H^2 \subseteq S + \mathcal{J}(D)$.

Corollary (4.4): A proper sub-module S of a multiplication R-module D is Nearly Semi-2-Absorbing if and only if $d^2K \subseteq S$ for K is a sub-module of D and $d \in D$, implies that either $dK \subseteq S + \mathcal{J}(D)$ or $d^2 \subseteq S + \mathcal{J}(D)$.

Remark (4.5): The residual of Nearly Semi-2-Absorbing sub-module not need to be Nearly Semi-2-Absorbing ideal of R. The following example illustrates this.

Example (4.6): The sub-module $S = \langle \bar{8} \rangle$ of the Z-module Z_{48} is Nearly-2-Absorbing by example (3.3), but $[S :_Z Z_{48}] = 8Z$ it's not Nearly-2-Absorbing ideal of Z, because $2^2.2 \in 8Z$ for $2 \in Z$ but $2.2 \notin 8Z + \mathcal{J}(Z) = 8Z + (0) = 8Z$.

The following propositions show that under certain condition that Nearly-2-Absorbing sub-module implies the residual is Nearly-2-Absorbing ideal and conversely.

Proposition (4.7): Let S be a proper sub-module of a multiplication projective R-module$D$. Then S is Nearly Semi-2-Absorbing sub-module of D if and only if $[S :_{\text{R}} D]$ is Nearly Semi-2-Absorbing ideal of R.

Proof: ⇒) Let $I^2r \subseteq [S :_{\text{R}} D]$ for $r \in$ R and some ideal I of R, hence $I^2rD \subseteq S$. But S is Nearly Semi-2-Absorbing sub-module of D, then by proposition (3.7) either $IrD \subseteq S + \mathcal{J}(D)$ or $I^2 \subseteq [S + \mathcal{J}(D) :_{\text{R}} D]$. As D is multiplication, then $S = [S :_{\text{R}} D]D$ and since D is projective multiplication, then by proposition (2.9) $\mathcal{J}(D) = \mathcal{J}(\text{R})D$. Thus either $IrD \subseteq [S :_{\text{R}} D]D + \mathcal{J}(\text{R})D$ or $I^2D \subseteq [S :_{\text{R}} D]D + \mathcal{J}(\text{R})D$, hence either $Ir \subseteq [S :_{\text{R}} D] + \mathcal{J}(\text{R})$ or $I^2 \subseteq [S :_{\text{R}} D] + \mathcal{J}(\text{R}) = \big[[S :_{\text{R}} D] + \mathcal{J}(\text{R}) : \text{R}\big]$. Therefore $[S :_{\text{R}} D]$ is Nearly Semi-2-Absorbing ideal of R.

⇐) Let $H^2 d \subseteq S$ for H is a sub-module of D, and $d \in D$, thus $H^2(d) \subseteq S$. As D is multiplication, then $H = ID$ and $(d) = \mathcal{J}D$ for some ideals I, $\mathcal{J}$ in Ɍ, that is $I^2\mathcal{J}D \subseteq S$, implies that $I^2\mathcal{J} \subseteq [S :_{\text{R}} D]$, but $[S :_{\text{R}} D]$ is Nearly Semi-2-Absorbing ideal of Ɍ, then either $I\mathcal{J} \subseteq [S :_{\text{R}} D] + \mathcal{J}(\text{R})$ or $I^2 \subseteq \left[[S :_{\text{R}} D] + \mathcal{J}(\text{R}) :_{\text{R}} \text{R}\right] = [S :_{\text{R}} D] + \mathcal{J}(\text{R})$, thus either $I\mathcal{J}D \subseteq [S :_{\text{R}} D]D + \mathcal{J}(\text{R})D$ or $I^2 D \subseteq [S :_{\text{R}} D]D + \mathcal{J}(\text{R})D$. Hence by proposition (2.9) either $I\mathcal{J}D \subseteq S + \mathcal{J}(D)$ or $I^2 D \subseteq S + \mathcal{J}(D)$, thus either $Hd \subseteq Ҡ + \mathcal{J}(D)$ or $H^2 \subseteq S + \mathcal{J}(D)$, hence S is Nearly Semi-2-Absorbing sub-module of D.

Proposition (4.8): Let S be a proper sub-module of a faithful multiplication Ɍ-module D. Then S is Nearly Semi-2-Absorbing sub-module of D if and only if $[S :_{\text{R}} D]$ is Nearly Semi-2-Absorbing ideal of Ɍ.

Proof: ⇒) Let $r^2\mathcal{J} \subseteq [S :_{\text{R}} D]$ for some ideal $\mathcal{J}$ of Ɍ and $r \in$ Ɍ, hence $r^2(\mathcal{J}D) \subseteq S$. But S is Nearly Semi-2-Absorbing sub-module, then by corollary (3.8) either $r(\mathcal{J}D) \subseteq S + \mathcal{J}(D)$ or $r^2 \in [S + \mathcal{J}(D) :_{\text{R}} D]$. As D is multiplication, then $S = [S :_{\text{R}} D]D$ and since D is faithful multiplication, then by proposition (2.11) $\mathcal{J}(D) = \mathcal{J}(\text{R})D$. Thus either $I(\mathcal{J}D) \subseteq [S :_{\text{R}} D]D + \mathcal{J}(\text{R})D$ or $r^2 D \subseteq [S :_{\text{R}} D]D + \mathcal{J}(\text{R})D$, thus either $I\mathcal{J} \subseteq [S :_{\text{R}} D] + \mathcal{J}(\text{R})$ or $r^2 \in [S :_{\text{R}} D] + \mathcal{J}(\text{R}) = \left[[S :_{\text{R}} D] + \mathcal{J}(\text{R}) : \text{R}\right]$. Hence $[S :_{\text{R}} D]$ is Nearly Semi-2-Absorbing ideal of Ɍ.

⇐) Let $H^2 L \subseteq S$ for H and L are a sub-modules of D. As D is a multiplication, then $H = ID$ and $L = \mathcal{J}D$ for some ideals I, $\mathcal{J}$ in Ɍ, that is $I^2\mathcal{J}D \subseteq S$, implies that $I^2\mathcal{J} \subseteq [S :_{\text{R}} D]$, but $[S :_{\text{R}} D]$ is Nearly Semi-2-Absorbing ideal of Ɍ, then either $I\mathcal{J} \subseteq [S :_{\text{R}} D] + \mathcal{J}(\text{R})$ or $I^2 \subseteq \left[[S :_{\text{R}} D] + \mathcal{J}(\text{R}) :_{\text{R}} \text{R}\right] = [S :_{\text{R}} D] + \mathcal{J}(\text{R})$, thus either $I\mathcal{J}D \subseteq [S :_{\text{R}} D]D + \mathcal{J}(\text{R})D$ or $I^2 D \subseteq [S :_{\text{R}} D]D + \mathcal{J}(\text{R})D$. Hence by proposition (2.11) either $I\mathcal{J}D \subseteq S + \mathcal{J}(D)$ or $I^2 D \subseteq S + \mathcal{J}(D)$, thus either $HL \subseteq S + \mathcal{J}(D)$ or $H^2 \subseteq S + \mathcal{J}(D)$. Thus by proposition (4.1) S is Nearly Semi-2-Absorbing.

Proposition (4.9): Let S be a proper sub-module of content multiplication Ɍ-module D. Then S is Nearly Semi-2-Absorbing sub-module of D if and only if $[S :_{\text{R}} D]$ is Nearly Semi-2-Absorbing ideal ofɌ.

Proof: ⇒) Let $r^2 c \in [S :_{\text{R}} D]$ for $r, c \in$ Ɍ, it follows that $r^2(cD) \subseteq S$. But S is Nearly Semi-2-Absorbing then by corollary (3.8) either $r(cD) \subseteq S + \mathcal{J}(D)$ or $r^2 \subseteq [S + \mathcal{J}(D) :_{\text{R}} D]$. As D is multiplication, then $S = [S :_{\text{R}} D]D$ and since D is a content, then by proposition (2.13) $\mathcal{J}(D) = \mathcal{J}(\text{R})D$. Thus either $rcD \subseteq [S :_{\text{R}} D]D + \mathcal{J}(\text{R})D$ or $r^2 D \subseteq [S :_{\text{R}} D]D + \mathcal{J}(\text{R})D$, hence either $rc \in [S :_{\text{R}} D] + \mathcal{J}(\text{R})$ or $r^2 \in [S :_{\text{R}} D] + \mathcal{J}(\text{R}) = \left[[S :_{\text{R}} D] + \mathcal{J}(\text{R}) : \text{R}\right]$. Hence $[S :_{\text{R}} D]$ is Nearly Semi-2-Absorbing ideal of Ɍ.

⇐) Let $r^2 L \subseteq S$ for $r \in$ Ɍ, L is a sub-module of D. As D is multiplication, then $L = \mathcal{J}D$ for some ideal $\mathcal{J}$ of Ɍ, that is $r^2\mathcal{J}D \subseteq S$, implies that $r^2\mathcal{J} \subseteq [S :_{\text{R}} D]$. But $[S :_{\text{R}} D]$ is Nearly Semi-2-Absorbing ideal of Ɍ, then either $r\mathcal{J} \subseteq [S :_{\text{R}} D] + \mathcal{J}(\text{R})$ or $r^2 \in \left[[S :_{\text{R}} D] + \mathcal{J}(\text{R}) :_{\text{R}} \text{R}\right] = [S :_{\text{R}} D] + \mathcal{J}(\text{R})$, thus either $r\mathcal{J}D \subseteq [S :_{\text{R}} D]D + \mathcal{J}(\text{R})D$ or $r^2 D \subseteq [S :_{\text{R}} D]D + \mathcal{J}(\text{R})D$. Since D is multiplication and content, then either $rL \subseteq S + \mathcal{J}(D)$ or $r^2 \in [S + \mathcal{J}(D) :_{\text{R}} D]$. Thus by corollary (3.8) S is Nearly Semi-2-Absorbing sub-module of D.

Proposition (4.10): Let S be a proper sub-module of a multiplication R-module D over a good ring R. Then S is Nearly Semi-2-Absorbing sub-module of D if and only if $[S:_{\mathrm{R}} D]$ is Nearly Semi-2-bsorbing ideal of R.

Proof: ⇒) Let $I^2\mathcal{J} \subseteq [S:_{\mathrm{R}} D]$ for I, $\mathcal{J} \subseteq$ R, it follows that $I^2\mathcal{J}D \subseteq S$. But S is Nearly Semi-2-Absorbing sub-module of D, then either $I\mathcal{J}D \subseteq S + \mathcal{J}(D)$ or $I^2 \subseteq [S + \mathcal{J}(D):_{\mathrm{R}} D]$. As D is multiplication, then $S = [S:_{\mathrm{R}} D]D$, and since R is a good ring, then by definition (2.14) $\mathcal{J}(D) = \mathcal{J}(\mathrm{R})D$. Thus either $I\mathcal{J}D \subseteq [S:_{\mathrm{R}} D]D + \mathcal{J}(\mathrm{R})D$ or $I^2D \subseteq [S:_{\mathrm{R}} D]D + \mathcal{J}(\mathrm{R})D$, hence either $I\mathcal{J} \subseteq [S:_{\mathrm{R}} D] + \mathcal{J}(\mathrm{R})$ or $I^2 \in [S:_{\mathrm{R}} D] + \mathcal{J}(\mathrm{R}) = \big[[S:_{\mathrm{R}} D] + \mathcal{J}(\mathrm{R}): \mathrm{R}\big]$. Hence $[S:_{\mathrm{R}} D]$ is Nearly Semi-2-Absorbing ideal of R.

⇐) Let $d^2L \subseteq S$ for $d \in D$, L is a sub-module of D. As D is multiplication, then $d = ID$ and $L = \mathcal{J}D$ for some ideals $I, \mathcal{J}$ in R, that is $I^2\mathcal{J}D \subseteq S$, implies that $I^2\mathcal{J} \subseteq [S:_{\mathrm{R}} D]$. But $[S:_{\mathrm{R}} D]$ is Nearly Semi-2-Absorbing ideal of R, then either $I\mathcal{J} \subseteq [S:_{\mathrm{R}} D] + \mathcal{J}(\mathrm{R})$ or $I^2 \in \big[[S:_{\mathrm{R}} D] + \mathcal{J}(\mathrm{R}):_{\mathrm{R}} \mathrm{R}\big] = [S:_{\mathrm{R}} D] + \mathcal{J}(\mathrm{R})$, thus either $I\mathcal{J}D \subseteq [S:_{\mathrm{R}} D]D + \mathcal{J}(\mathrm{R})D$ or $I^2D \subseteq [S:_{\mathrm{R}} D]D + \mathcal{J}(\mathrm{R})D$. Since D is multiplication and R is good ring then either $dL \subseteq S + \mathcal{J}(D)$ or $d^2 \in [S + \mathcal{J}(D):_{\mathrm{R}} D]$. Therefore S is Nearly Semi-2-Absorbing sub-module of D.

Corollary (4.11): Let S be a proper sub-module of a multiplication R-module D over Artinian ring R. Then S is Nearly Semi-2-Absorbing sub-module of D if and only if $[S:_{\mathrm{R}} D]$ is Nearly Semi-2-bsorbing ideal of R.

Proposition (4.12): Let S be a proper sub-module of a multiplication R-module D over a local ring R. Then S is the Nearly Semi-2-Absorbing sub-module of D if and only if $[S:_{\mathrm{R}} D]$ is the Nearly Semi-2-absorbing ideal of R.

Proof: In the same way as the proposition(4.10).

The following propositions are characterized by Nearly-2-Absorbing ideals by a special kind of Nearly Semi-2-Absorbing sub-modules.

Proposition (4.13): Let D be a finitely-generated multiplication projective R-module, and P is an ideal of R with $ann_{\mathrm{R}}(D) \subseteq P$. Then P is the Nearly Semi-2-Absorbing ideal of R if and only if PD is the Nearly Semi-2-Absorbing sub-module of D.

Proof: ⇒) Let $H^2d \subseteq PD$, for H is a sub-module of D and $d \in D$, that is $H^2(d) \subseteq PD$. As D is multiplication, then $H^2 = I^2D$ and $(d) = \mathcal{J}D$ for some ideals $I, \mathcal{J}$ in R, that is $I^2\mathcal{J}D \subseteq PD$. But D is a finitely-generated multiplication R-module then by proposition (2.18) $I^2\mathcal{J} \subseteq P + ann_{\mathrm{R}}(D)$, but $ann_{\mathrm{R}}(D) \subseteq P$, implies that $P + ann_{\mathrm{R}}(D) = P$, thus $I^2\mathcal{J} \subseteq P$. Now, by assumption P is Nearly Semi-2-Absorbing ideal of R, either $I\mathcal{J} \subseteq P + \mathcal{J}(\mathrm{R})$ or $I^2 \subseteq [P + \mathcal{J}(\mathrm{R}):_{\mathrm{R}} \mathrm{R}] = P + \mathcal{J}(\mathrm{R})$, it follows that either $I\mathcal{J}D \subseteq PD + \mathcal{J}(\mathrm{R})D$ or $I^2D \subseteq PD + \mathcal{J}(\mathrm{R})D$. Since D is projective then by proposition (2.9) $\mathcal{J}(D) = \mathcal{J}(\mathrm{R})D$, it follows either $H(d) \subseteq PD + \mathcal{J}(D)$ or $H^2 \subseteq [PD + \mathcal{J}(D):_{\mathrm{R}} D]$. Hence PD is Nearly Semi-2-Absorbing sub-module of D.

⇐) Let $r^2I \subseteq P$, for I is an ideal ofR and $r \in$ R, implies that $r^2(ID) \subseteq PD$. But PD is Nearly Semi-2-Absorbing sub-module of D, then either $r(ID) \subseteq PD + \mathcal{J}(D)$ or $r^2D \subseteq PD + \mathcal{J}(D)$. But D is a projective then $\mathcal{J}(D) = \mathcal{J}(\mathrm{R})D$. Thus, either $rID \subseteq PD + \mathcal{J}(\mathrm{R})D$ or $r^2D \subseteq PD + \mathcal{J}(\mathrm{R})D$, it follows that either $rI \subseteq P + \mathcal{J}(\text{R})$ or $r^2 \in P + \mathcal{J}(\text{R}) = [P + \mathcal{J}(\text{R}):_{\text{R}} \text{R}]$. Hence by corollary (3.8) P is Nearly Semi-2-Absorbing ideal of R.

Proposition (4.14): Let D be a faithful finitely-generated multiplication R-module and P is an ideal of R. Then P is Nearly Semi-2-Absorbing ideal of R if and only if PD is Nearly Semi-2-Absorbing sub-module of D.

Proof: ⇒) Let $d^2K \subseteq PD$, for $d \in D$ and K is a sub-module of D, it follows that $(d^2)K \subseteq PD$. As D is multiplication, then $(d)^2 = I^2D$ and $K = \mathcal{J}D$for some ideals $I, \mathcal{J}$ in R, that is $I^2\mathcal{J}D \subseteq PD$. But D is a finitely-generated multiplication R-module then by proposition (2.18) $I^2\mathcal{J} \subseteq P + ann_{\text{R}}(D)$ and since D is faithful, then $ann_{\text{R}}(D) = (0)$, implies that $P + ann_{\text{R}}(D) = P$, hence $I^2\mathcal{J} \subseteq P$. But P is Nearly Semi-2-Absorbing ideal of R $+\mathcal{J}(D)$or $d^2 \subseteq PD + \mathcal{J}(D)$. Therefore PD is Nearly Semi-2-Absorbing sub-module of D.

⇐) Let $r^2c \in P$, for $r, c \in \text{R}$, implies that $r^2(cD) \subseteq PD$. Since PD is Nearly Semi-2-Absorbing sub-module of D, then either $r(cD) \subseteq PD + \mathcal{J}(D)$ or $r^2 \in [PD + \mathcal{J}(D):_{\text{R}} D]$. That is either $rcD \subseteq PD + \mathcal{J}(D)$ or $r^2D \subseteq PD + \mathcal{J}(D)$. But D is faithful multiplication, then either $rcD \subseteq PD + \mathcal{J}(\text{R})D$ or $r^2D \subseteq PD + \mathcal{J}(\text{R})D$, it follows either $rc \in P + \mathcal{J}(\text{R})$ or $r^2 \in P + \mathcal{J}(\text{R}) = [P + \mathcal{J}(\text{R}):_{\text{R}} \text{R}]$. Hence P is Nearly Semi-2-Absorbing ideal of R.

Proposition (4.15): Let D be a finitely-generated multiplication content R-module and P is an ideal of R with $ann_{\text{R}}(D) \subseteq P$. Then P is Nearly Semi-2-Absorbing ideal of R if and only if PD is Nearly Semi-2-Absorbing sub-module of D.

Proof: Similar to the proposition(4.13).

Proposition (4.16): Let D be a finitely-generated multiplication module over good ring R and P is an ideal of R such that $ann_{\text{R}}(D) \subseteq P$. Then P is Nearly Semi-2-Absorbing ideal of R if and only if PD is Nearly Semi-2-Absorbing sub-module of D.

Proof: Directly from proposition(4.13) and definition(2.14).

Corollary (4.17): Let D be a finitely-generated multiplication module over Artinian ring R and P is an ideal of R such that $ann_{\text{R}}(D) \subseteq P$. Then P is Nearly Semi-2-Absorbing ideal of R if and only if PD is Nearly Semi-2-Absorbing sub-module of D.

Proposition (4.18): Let D be a finitely-generated multiplication module over a local ring R and P is an ideal of R such that $ann_{\text{R}}(D) \subseteq P$. Then P is Nearly Semi-2-Absorbing ideal of R if and only if PD is Nearly Semi-2-Absorbing sub-module of D.

Proof: Directly from proposition(4.13) and proposition(2.16).

From propositions (4.7) and (4.13), we obtain the following result.

Corollary (4.19): Let S be a proper sub-module of a finitely-generated multiplication projective R-module D such that $ann_{\text{R}}(D) \subseteq [S:_{\text{R}} D]$**.** Then the following is equivalent.

1. S is Nearly Semi-2-Absorbing sub-module of D.

2. $[S:_{\mathrm{R}} D]$ is Nearly Semi-2-Absorbing ideal of R.

3. $S = PD$ for some Nearly Semi-2-Absorbing ideal P of R with $ann_{\mathrm{R}}(D) \subseteq P$.

As a direct consequence of proposition(4.8) and (4.14) we get this corollary.

Corollary (4.20): Let S be a proper sub-module of a faithful finitely-generated multiplication R-module D**.** Then the following is equivalent.

1. S is Nearly Semi-2-Absorbing sub-module of D.

2. $[S:_{\mathrm{R}} D]$ is Nearly Semi-2-Absorbing ideal of R.

3. $S = PD$ for some Nearly Semi-2-Absorbing ideal P of R.

This corollary follows directly from propositions (4.9) and (4.15).

Corollary (4.21): Let S be a proper sub-module of a finitely-generated multiplication content R-module D such that $ann_{\mathrm{R}}(D) \subseteq [S:_{\mathrm{R}} D]$**.** Then the following is equivalent.

1. S is Nearly Semi-2-Absorbing sub-module of D.

2. $[S:_{\mathrm{R}} D]$ is Nearly Semi-2-Absorbing ideal of R.

3. $S = PD$ for some Nearly Semi-2-Absorbing ideal P of R with $ann_{\mathrm{R}}(D) \subseteq P$.

From proposition(4.10) and (4.16) we get this corollary.

Corollary (4.22): Let S be a proper sub-module of a finitely-generated multiplication R-module D over a good ring R such that $ann_{\mathrm{R}}(D) \subseteq [S:_{\mathrm{R}} D]$**.** Then the following is equivalent.

1. S is Nearly Semi-2-Absorbing sub-module of D.

2. $[S:_{\mathrm{R}} D]$ is Nearly Semi-2-Absorbing ideal of R.

3. $S = PD$ for some Nearly Semi-2-Absorbing ideal P of R with $ann_{\mathrm{R}}(D) \subseteq P$.

From corollary(4.11) and (4.17) we get this corollary.

Corollary (4.23): Let S be a proper sub-module of a finitely-generated multiplication R-module D over Artinian ring R such that $ann_{\mathrm{R}}(D) \subseteq [S:_{\mathrm{R}} D]$**.** Then the following is equivalent.

1. S is Nearly Semi-2-Absorbing sub-module of D.

2. $[S:_{\mathrm{R}} D]$ is Nearly Semi-2-Absorbing ideal of R.

3. $S = PD$ for some Nearly Semi-2-Absorbing ideal P of R with $ann_{\mathrm{R}}(D) \subseteq P$.

From proposition(4.12) and (4.18) we get this corollary.

Corollary (4.24): Let S be a proper sub-module of a finitely-generated multiplication content R-module D over local ring R such that $ann_{\mathrm{R}}(D) \subseteq [S:_{\mathrm{R}} D]$**.** Then the following is equivalent.

1. S is Nearly Semi-2-Absorbing sub-module of D.

2. $[S:_{\mathrm{R}} D]$ is Nearly Semi-2-Absorbing ideal of R.

3. $S = PD$ for some Nearly Semi-2-Absorbing ideal P of R with $ann_{\mathrm{R}}(D) \subseteq P$.

At last we prove that in the following propositions the two concepts (Semi-2-Absorbing and Nearly Semi-2-Absorbing) sub-modules are equivalent under certain condition.

Proposition (4.25): Let S be a proper sub-module of an R-module D, such that $\mathcal{J}(D) \subseteq S$. Then S is Semi-2-Absorbing sub-module of D if and only if S is Nearly Semi-2-Absorbing.

Proof: ⇒) By remark(3.4).

⇐) Since $\mathcal{J}(D) \subseteq S$, then $\mathcal{J}(D) + S = S$. So the proof is direct.

Proposition (4.26): Let D be a module over a v-ring R, and S be a proper sub-module of D. Then S is Semi-2-Absorbing sub-module of D if and only if S is Nearly Semi-2-Absorbing.

Proof: ⇒) By remark(3.4).

⇐) Since D is a module over a v-ring R, then $\mathcal{J}(D) = 0$. So the proof is direct.

Proposition (4.27): Let S be a proper sub-module of a regular R-module D. Then S is Semi-2-Absorbing sub-module of D if and only if S is Nearly Semi-2-Absorbing.

Proof: ⇒) By remark(3.4).

⇐) Since D is regular, then $\mathcal{J}(D) = 0$. So the proof is direct.

As a direct consequence of proposition(4.25) we get this corollary.

Corollary (4.28): Let S be a proper sub-module of an R-module D, such that $\mathcal{J}\left(\frac{D}{S}\right) = 0$. Then S is Semi-2-Absorbing sub-module of D if and only if S is Nearly Semi-2-Absorbing.

Finally in this paper we remark that all rings are commutative rings with identity and all modules are left R-modules with a unitary.

Conclusion

Among the most important results are:

- Every 2-Absorbing (Semi-2-Absorbing) sub-module is Nearly Semi-2-Absorbing, but contrariwise isn't true in general.
- A proper sub-module S of an R-module D is Nearly Semi-2-Absorbing sub-module of D if and only if for any $r \in \mathrm{R}$ such that $r^2 \notin [S + \mathcal{J}(D) :_{\mathrm{R}} D]$, then $[S:_D r^2] \subseteq [S + \mathcal{J}(D):_D r]$.

- A proper sub-module S of a multiplication Ɍ-module D is Nearly Semi-2-Absorbing if and only if $H^2K \subseteq S$ for H and K are sub-modules of D, implies that either $HK \subseteq S + \mathcal{J}(D)$ or $H^2 \subseteq S + \mathcal{J}(D)$

- In content multiplication Ɍ-module D, a proper sub-module S is Nearly Semi-2-Absorbing sub-module of D if and only if $[S:_{Ɍ} D]$ is Nearly Semi-2-Absorbing ideal ofɌ.

- Let S be a proper sub-module of a finitely-generated multiplication content Ɍ-module D such that $ann_{Ɍ}(D) \subseteq [S:_{Ɍ} D]$**.** Then the following is equivalent.

  1. S is Nearly Semi-2-Absorbing sub-module of D.

  2. $[S:_{Ɍ} D]$ is Nearly Semi-2-Absorbing ideal of Ɍ.

  3. $S = PD$ for some Nearly Semi-2-Absorbing ideal P of Ɍ with $ann_{Ɍ}(D) \subseteq P$.

- In regular Ɍ-module D, a proper sub-module S is Semi-2-Absorbing if and only if S is Nearly Semi-2-Absorbing sub-module of D.

- Let D be an Ɍ-module over a v-ring and S be proper sub-module of D. Then S is Semi-2-Absorbing if and only if S is Nearly Semi-2-Absorbing sub-module of D.

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حول المقاسات الجزئية شبه الممتصة تقريبا

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الخلاصة:

في هذا البحث، قلنا ان الحلقة تبادلية مع عنصر محايد و D تكون مقاساً ايسرياً وحدوياً. إذ قدمت المقاسات الجزئية شبه الممتصة تقريبا كتعميم جديد للمقاسات الجزئية الممتصة وشبه الممتصة. اعطينا العديد من الخصائص الاساسية والمتكافئات والأمثلة لهذا المفهوم. فضلا عن ذلك تم تحديد العديد من الخصائص للمقاسات الجزئية شبه الممتصة تقريبًا في صف المقاسات الجدائية. علاوة على ذلك اوضحنا من الأمثلة، إن ما تبقى من المقاسات الجزئية شبه الممتصة تقريبًا لا يحتاج إلى أن تكون مُثاليا شبه ممتص تقريبًا ل R. لذلك برهنا هذا الحالة في بعض انواع المقاسات.