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The Long-Time Behavior of Stochastic Prey-Predator Model with the Hide-and-Escape Effect for Prey and Predation Skill Augmentation for Predator

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Abstract

In this paper, a comprehensive investigation was conducted of the stochastic prey-predator model combined with the functional and numerical response of the Holling Type II under the influence of the Hide-and-Escape Effect (HEE) for the prey and Predation Skill Augmentation (PSA) for the predator. Initially, the significance of novel effects was elucidated, along with their formulation and integration into the fundamental model. A comprehensive analysis was performed on the proposed model. Theoretically, it has been proven that there is a unique solution in the positive region, and this solution is a global solution. The permanent and extinct conditions of species were also studied in the model. Numerically, the model was simulated using MATLAB, and the resultant theoretical findings were validated.

Introduction:

Exploring the dynamics between competing species is of great importance in various disciplines of applied sciences. These dynamics have urged researchers in many fields to conduct research on them, especially in physics, chemistry, and ecological studies, by modeling these dynamics using differential equations [1]. Regarding ecological sciences, the basic model in this field was presented by the scientists Lotka [2] and Volterra [3], who created a model that depicts the competitive dynamics between prey and predator. Although their basic model was simplistic, it was considered a great achievement in depicting the complex interactions between prey and predator.

The previous model was developed by including many additions to it, the most important of which are numerical and functional responses. The functional response is defined as the change in the density of prey resulting from the predation process [4]. According to this definition, many functional responses have been modeled, the most prominent of which are the three types that the scientist Holling added to the prey-predator model [5-6]. On the other hand, the numerical response is defined as the change in the density of the predator resulting from the predation process. Integrating the functional and numerical response into the prey-predator model led to an increase in the accuracy of the models in representing the real interactions between the prey and the predator [7].

The deterministic approach is the basis for modeling this type of interaction, but the growing interest in this topic in order to understand the oscillating dynamics occurring in real environments has prompted many researchers to include white noise in any variable within the model. Integrating white noise into differential equations produces stochastic differential equations. This type of equation allows to increase the accuracy of representing environmental fluctuations that occur in ecosystems that the deterministic equations have a lack in represent it [8].

In recent years, many external effects have been added to the stochastic predator-prey model. In 2017, Zhang investigated the global dynamics of a stochastic model with non-constant mortality rates, and he established the sufficient conditions for extinction, persistence, and periodic solutions [9]. In 2018, Liu et al. introduced a stochastic model with stage structure for predators, investigating ergodic stationary distribution and extinction conditions [10]. In 2020, the dynamics of the stochastic model were analyzed by Gokila et al. under the influence of the presence of a disease in the prey, and the results were clarified by conducting numerical simulations of the model and comparing them with theoretical results [11]. Also in the same year, Rihan and Alsakajiand studied the random model combined with the effect of cooperative hunting on predators, and they reached the conditions that must be met in the model in order for its continued coexistence [12]. Shi et al. (2021) studied a stochastic model with stage structure and refuge for prey [13]. Also, in 2021, the stochastic model was developed by Belabbas et al. by imposing the presence of protected areas for prey and studying the dynamics of the system under the presence of this effect [14]. In 2022, Das et al. conducted a study on the long-term behavior of the stochastic model under the influence of the collective defence of the prey, and it was concluded that the model coexists under certain conditions [15]. In 2023, Rao and Kang studied the model under the influence of prey refuge and predation fear effects, and it was concluded that the presence of these influences in reasonable proportions leads to the stability of the model [16]. In 2024, Liu studied the dynamics of coexistence and extinction in a stochastic prey-predator system under the effect of switching [17].

In this study we aim to add two new effects to the stochastic model. Competing animals have many behaviors. Such as Predation Skill Augmentation (PSA), which is defined as the predator's ability to develop his predation skill through the accumulation of experience [18]. The second is Hide-and-Escape Effect (HEE), which describes the ability of the prey to escape from the predator or find an opportunity to hide from it, which reduces the possibility of an encounter between the prey and the predator [19].

Mathematically, the effect of PSA can be represented by the function,

$$j(1-e^{-y}),$$

where $y=y(t)$ denotes predator density over time t, and j is a positive parameter that determines the proportion of predators that are able to improve their predation skills. The value of this function ranges from 0 to j, assuming that the predator density is non-negative ($y\geq 0$). This function indicates that the predation skill improves with the increase in the density of predators until it reaches its maximum, which is j, where the exponential term e^{-y} ensures that the increase in skill occurs gradually over time, which reflects the realistic behavior in the development of the predation skill of predators in a natural environment.

The effect of HEE was mathematically modelled using the cosine function,

$$m\big(1+cos(x)\big),$$

which fluctuates based on the availability of opportunities that allow the prey to escape or hide from predators, where $x=x(t)$ represents the density of prey over time t and m is a positive parameter that shows the percentage of prey that are able to successfully escape or hide. The periodic nature of the cosine function is well able to represent this behavior because its values range between 1 and -1, and as a result, the effect of the added function will range between 0 and $2m$.

This mathematical formulation acknowledges the cyclical nature of prey behavior in response to predation risk. When conditions are favorable for hiding or escaping, a significant portion of the prey population might successfully avoid predators, represented by the function's peak value at $2m$. Conversely, when conditions are less conducive to such behaviors, the effectiveness of these strategies decreases, illustrated by the function's minimum value at 0. The range of this function, $[0,2m]$, encapsulates the entire spectrum of the HEE's variability, mirroring the dynamic and fluctuating potential for prey to evade predation across different environmental conditions and over time.

Integrating the "Predation Skill Augmentation" (PSA) effect and the "Hide-and-Escape Effect" (HEE) into a Holling Type II functional and numerical response prey-predator model, along with incorporating standard Brownian motions $\mathcal{B}_1(t)$ and $\mathcal{B}_2(t)$ affecting the growth rates of prey and predator populations respectively, yields a comprehensive stochastic model. This model captures the dynamics of prey-predator interactions with a nuanced consideration of natural behaviors and environmental randomness. The resulting stochastic differential equations (SDEs) for the model can be represents as follows:

$$\begin{aligned} dx(t) &= x(t)\left[r-\frac{r}{k}x(t)-\frac{ay(t)}{1+ahx(t)}-j\big(1-e^{-y(t)}\big)\right]dt+\ell_1 x(t)d\mathcal{B}_1(t),\\ dy(t) &= y(t)\left[\frac{eax(t)}{1+ahx(t)}-\frac{eay(t)}{1+ahx(t)}-m\Big(1+cos\big(x(t)\big)\Big)-u\right]dt \\ &\quad +\ell_2 y(t)d\mathcal{B}_2(t). \end{aligned} \tag{1}$$

Where the components of model (1) are illustrated in Tab. (1),

Table 1: The component of system (1).

Component	Ecological Meaning
$x(t)$	Density of the prey population
$y(t)$	Density of the predator population
r	Growth rate of the prey population
a	Rate of predation
u	Decay rate of the predator population
h	Half-saturation constant indicating the prey density at which predation rate is half its maximum
e	Efficiency with which consumed prey is converted into predator biomass
k	Carrying capacity of the environment for the prey population
j	Proportion of predators capable of enhancing their predation skills
m	Proportion of prey that have the opportunity to escape or hide from predators
$\mathcal{B}_1(t)$ and $\mathcal{B}_2(t)$	Independent standard Brownian motions representing random environmental fluctuations
ℓ_1 and ℓ_2	Standard deviations of the intensities of environmental white noise affecting prey and predator populations, respectively

Given the non-linear nature of the Hide-and-Escape Effect (HEE) and Predation Skill Augmentation (PSA), which complicates analytical treatment, we have employed certain approximations to ease the theoretical analysis of this paper. since PSA is bounded by $[0, j]$ in R_+^2, and HEE lies in $[0,2m]$, then let we assume that $\gamma = j\left(1 - e^{-y(t)}\right)$ and $\delta = m\left(1 + cos\big(x(t)\big)\right)$. Also, assume that $x(t) = x, y(t) = y, \mathcal{B}_1(t) = \mathcal{B}_1$, and $\mathcal{B}_2(t) = \mathcal{B}_2$. Then model (1) can be expressed by:

$$\begin{aligned} dx(t) &= x(t)\left[r - \frac{r}{k}x(t) - \frac{ay(t)}{1 + ahx(t)} - \gamma\right]dt + \ell_1 x(t) d\mathcal{B}_1(t), \\ dy(t) &= y(t)\left[\frac{eax(t)}{1 + ahx(t)} - \frac{eay(t)}{1 + ahx(t)} - \delta - u\right]dt + \ell_2 y(t) d\mathcal{B}_2(t). \end{aligned} \tag{2}$$

The rest of the paper is organized as follows: Section 2 presents the existence and uniqueness of positive solution, where Section 3 is covers the long-time behavior of model (1), Section 4 explores the numerical simulations, finally, Section 5 provides the conclusion of the paper.

Existence and uniqueness of positive solution

In this section, we focus on the existence of positive solutions due to their biological meanings. Furthermore, for a stochastic differential equation to possess a unique global solution, meaning it does not explode in a finite time, for any given initial condition, the coefficients of the equation usually need to fulfill the linear growth condition and the local Lipschitz condition. However, the coefficients of system (2) do not meet either the linear growth condition or are locally Lipschitz continuous [20]. In this section, by applying a change of variables and utilizing the comparison theorem of stochastic equations (see [21]), we demonstrate that there exists a unique positive solution with a positive initial value for system (2), and this solution is global.

Theorem 1: *For any* $\left(x(0), y(0)\right) \in R_+^2$, *then system (2) has a local solution* (x, y) *which is unique and positive for* $t \in [0, \mathscr{t}_E]$, *where* $\mathscr{t}_E$ *is the explosion time.*

Proof: By applying the transformation, $x = e^P$ and $y = e^H$, which implies that $P = lnx$ and $H = lny$, and employing Ito's formula on system (2), we obtain the transformed system:

$$\begin{aligned} dP &= \left[r\left(1 - \frac{e^P}{k}\right) - \frac{ae^H}{1 + ahe^P} - \gamma - 0.5{\ell_2}^2\right]dt + \ell_1 d\mathcal{B}_1, \\ dH &= \left[\frac{eae^P}{1 + ahe^P} - \frac{eae^H}{1 + ahe^P} - u - \delta - 0.5{\ell_2}^2\right]dt + \ell_2 d\mathcal{B}_2, \end{aligned} \tag{3}$$

with initial conditions where $P(0) = lnx_0$ and $H(0) = lny_0$. It is evident that the transformed system (3) satisfies the conditions required for a unique, local, and positive solution (x, y) that originates from an interior point in R_+^2 over the interval $[0, \mathscr{t}_E]$, where $\mathscr{t}_E$ is a finite positive real number and the partial derivatives in P and H remain bounded.

After proving that the system (2) contains a local solution, we turn our attention to broadening this solution to a global scale. Our goal is to ascertain that the dynamic interactions modeled are perpetual. A crucial step in this validation process involves proving that the explosion time $\mathscr{t}_E = \infty$. The theorem below solidifies this concept. But we first we will give the definition of Itô formula,

Definition 1 [22]: Suppose that $X(\mathscr{t})$ be a stochastic process satisfies,

$$dX(t) = adt + bd\mathcal{B}(t)$$

And let $\mathcal{R} = f\left(t, X(t)\right)$, where f is a function of t and $X(t)$. Then, the derivative of Then the Itô formula is,

$$d\mathcal{R} = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}dX + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}(dX)^2.$$

$$d\mathcal{R} = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}\left(adt + bd\mathcal{B}(t)\right) + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}\left(adt + bd\mathcal{B}(t)\right)^2.$$

$$d\mathcal{R} = \frac{\partial f}{\partial t}dt + a\frac{\partial f}{\partial X}d\mathscr{t} + b\frac{\partial f}{\partial X}d\mathcal{B}(t) + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}\left(a^2(dt)^2 + abdtd\mathcal{B}(t) + b^2\left(d\mathcal{B}(t)\right)^2\right).$$

According to the fact that $(dt)^2 = dtd\mathcal{B}(t) = 0$, and $\left(d\mathcal{B}(t)\right)^2 = dt$, Then the Itô formula is,

$$d\mathcal{R} = \left(\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial X} + \frac{b^2}{2}\frac{\partial^2 f}{\partial X^2}\right)dt + b\frac{\partial f}{\partial X}d\mathcal{B}(t).$$

Theorem 2: *Assume* (x, y) *be a positive solution for the system (2) when* $t \in [0, \mathscr{t}_E]$, *then there exist* $\overline{X}, \underline{X}, \overline{Y}$ *and* $\underline{Y}$ *such that,* $\underline{X} \le x \le \overline{X}$, *and* $\underline{Y} \le y \le \overline{Y}$, $\forall t \ge 0$. *Which indicate* (x, y) *is global solution and* $\mathscr{t}_E = \infty$.

Proof: if we show that $\mathscr{t}_E = \infty$ then we can say (x, y) is global solution, from the prey equation in system (2) we have,

$$dx \le rxdt + \ell_1 xd\mathcal{B}_1$$

Let $\overline{X}$ be the solution of

$$\begin{cases} d\overline{X} \ = r\overline{X}dt + \ell_1\overline{X}d\mathcal{B}_1 \\ \qquad \overline{X}(0) = \overline{X}_0 \end{cases}, \tag{4}$$

To find $\overline{X}$, by dividing both sides of (4) by $\overline{X}$,

$$\frac{d\overline{X}}{\overline{X}} = rdt + \ell_1 d\mathcal{B}_1. \tag{5}$$

By integrating both sides of (5), we get,

$$ln\overline{X} = rt + \ell_1\mathcal{B}_1 + c, \tag{6}$$

where c is function of t, then (6) can be written as follows,

$$\overline{X} = e^{rt+\ell_1\mathcal{B}_1+c(t)}. \tag{7}$$

To find $c(t)$, using Definition 1, Eq. (7) can be written,

$$\overline{dX} = \big(r + \acute{c}(t)\big)e^{rt+\ell_1\mathcal{B}_1+c(t)} + \ell_1 e^{rt+\ell_1\mathcal{B}_1+c(t)}d\mathcal{B}_1 + \frac{(\ell_1)^2}{2}e^{rt+\ell_1\mathcal{B}_1+c(t)}(d\mathcal{B}_1)^2,$$

$$d\overline{X} \ = \big(r + \acute{c}(t)\big)\overline{X}dt + \ell_1\overline{X}d\mathcal{B}_1 + \frac{(\ell_1)^2}{2}\overline{X}(d\mathcal{B}_1)^2,$$

$$d\overline{X} \ = \left(r + \acute{c}(t) + \frac{(\ell_1)^2}{2}\right)\overline{X}dt + \ell_1\overline{X}d\mathcal{B}_1. \tag{8}$$

By comparing (8) and (4), we have,

$$\acute{c}(t) = -\frac{(\ell_1)^2}{2},$$

$$c(t) = -\frac{(\ell_1)^2}{2}t + k^* \tag{9}$$

By substituting (9) in (7) we get,

$$\overline{X} = e^{rt+\ell_1\mathcal{B}_1-\frac{(\ell_1)^2}{2}t+k^*} = \overline{X}_0 e^{\left(r-\frac{(\ell_1)^2}{2}\right)t+\ell_1\mathcal{B}_1}.$$

Where $\overline{X}_0 = e^{k^*}$ after substituting $t = 0$.

Then we have $x \leq \overline{X}$, for all $t \in [0, t_E]$.

Now, for predator equation we have,

$$dy \geq (-eay^2 - uy - \delta y)dt + \ell_2 y d\mathcal{B}_2$$

Let $\underline{Y}$ be the solution of,

$$\begin{cases} d\underline{Y} = \big(-ea\underline{Y} - u - \delta\big)\underline{Y}dt + \ell_2\underline{Y}d\mathcal{B}_2 \\ \qquad \underline{Y}(0) = \underline{Y}_0 \end{cases}. \tag{10}$$

To find $\underline{Y}$, we will integrate both sides of (10) using the integrating factor $\imath = e^{-\int_0^t g(s)d\mathcal{B}_2(s)+\frac{1}{2}\int_0^t g^2(s)ds}$, where $g(t) = \ell_2$. Then, we have,

$$\imath = e^{-\int_0^t \ell_2 d\mathcal{B}_2(s)+\frac{1}{2}\int_0^t(\ell_2)^2 ds} = e^{-\ell_2\mathcal{B}_2+\frac{1}{2}(\ell_2)^2 t}. \tag{11}$$

By using Ito formula, we have

$$d\imath = \imath_t dt + \imath_{\mathcal{B}_2}d\mathcal{B}_2 + \frac{1}{2}\imath_{\mathcal{B}_2\mathcal{B}_2}(d\mathcal{B}_2)^2,$$

$$d\imath = \frac{1}{2}(\ell_2)^2\imath dt - \ell_2\imath d\mathcal{B}_2 + \frac{1}{2}(\ell_2)^2\imath dt,$$

$$d\imath = (\ell_2)^2\imath dt - \ell_2\imath d\mathcal{B}_2. \tag{12}$$

Now by multiplying (12) by (10),

$$d\underline{Y}d\imath = -(\ell_2)^2\underline{Y}\imath d\mathcal{B}_2.$$

Also, multiplying (11) by (10) and rearrange the result we get,

$$\imath d\underline{Y} - \imath\ell_2\underline{Y}d\mathcal{B}_2 = \imath\big(-ea\underline{Y} - u - \delta\big)\underline{Y}dt. \tag{13}$$

Consequently, after adding and subtracting $-(\ell_2)^2\underline{Y}i d\mathcal{B}_2$ to the left side of (13) we get,

$$i d\underline{Y} - i\ell_2\underline{Y}d\mathcal{B}_2 + (\ell_2)^2\underline{Y}i d\mathcal{B}_2 - (\ell_2)^2\underline{Y}i d\mathcal{B}_2 = i\big(-ea\underline{Y} - u - \delta\big)\underline{Y}dt,$$

$$i d\underline{Y} + \underline{Y}di + d\underline{Y}di = i\big(-ea\underline{Y} - u - \delta\big)\underline{Y}dt,$$

$$d\big(i\underline{Y}\big) = i\big(-ea\underline{Y} - u - \delta\big)\underline{Y}dt, \tag{14}$$

To solve Eq. (14) we use the following transformation $Y^* = i\underline{Y}$ and $\underline{Y} = \frac{Y^*}{i}$. Then,

$$dY^* = i\left(-ea\frac{Y^*}{i} - u - \delta\right)\frac{Y^*}{i}dt,$$

$$dY^* = \left(-\frac{ea}{i}Y^{*2} - (u+\delta)Y^*\right)dt,$$

$$\frac{dY^*}{dt} + (u+\delta)Y^* = -\frac{ea}{i}Y^{*2},$$

$$\frac{1}{Y^{*2}}\frac{dY^*}{dt} + \frac{1}{Y^*}(u+\delta) = -\frac{ea}{i}. \tag{15}$$

Let $Z = \frac{1}{Y^*}$, then,

$$\frac{dZ}{dY^*} = \frac{-1}{Y^{*2}}.$$

Where,

$$\frac{dZ}{dt} = \frac{dZ}{dY^*}\cdot\frac{dY^*}{dt},$$

Which indicates,

$$\frac{dZ}{dt} = \frac{-1}{Y^{*2}}\cdot\frac{dY^*}{dt}. \tag{16}$$

By substituting (16) in (15) after multiplying it by (-1) we get,

$$\frac{dZ}{dt} - (u+\delta)Z = \frac{ea}{i}. \tag{17}$$

The integrating factor for (17) is,

$$\mu = e^{\int_0^t -(u+\delta)ds} = e^{-(u+\delta)t}.$$

By multiplying the integrating factor by (17) and integrate both sides from 0 to t, it can be obtained,

$$\mu Z|_0^t = \int_0^t \frac{ea}{i}e^{-(u+\delta)s}ds.$$

Consequently,

$$e^{-(u+\delta)t}Z = Z_0 + ea\int_0^t e^{\ell_2\mathcal{B}_2(s)-\frac{1}{2}(\ell_2)^2 s}\,e^{-(u+\delta)S}ds,$$

$$e^{-(u+\delta)t}\frac{1}{Y^*} = \frac{1}{Y_0^*} + ea\int_0^t e^{-\left((u+\delta)+\frac{1}{2}(\ell_2)^2\right)s+\ell_2\mathcal{B}_2(s)}\,ds,$$

$$e^{-(u+\delta)t}\frac{1}{i\underline{Y}} = \frac{1}{\underline{Y_0}} + ea\int_0^t e^{-\left((u+\delta)+\frac{1}{2}(\ell_2)^2\right)s+\ell_2\mathcal{B}_2(s)}\,ds,$$

$$e^{-(u+\delta)t}e^{+\ell_2\mathcal{B}_2-\frac{1}{2}(\ell_2)^2 t}\frac{1}{\underline{Y}} = \frac{1}{\underline{Y_0}} + ea\int_0^t e^{-\left((u+\delta)+\frac{1}{2}(\ell_2)^2\right)s+\ell_2\mathcal{B}_2(s)}\,ds,$$

Hence, the solution is,

$$\underline{Y} = \frac{e^{-\left((u+\delta)+\frac{1}{2}(\ell_2)^2\right)t+\ell_2\mathcal{B}_2}}{\frac{1}{\underline{Y}_0} + ea\int_0^t e^{-\left((u+\delta)+\frac{1}{2}(\ell_2)^2\right)s+\ell_2\mathcal{B}_2(s)}\,ds}.$$

Then we have, $y \geq \underline{Y}$, for all $t \in [0, t_E]$.

Also, for predator we have,

$$dy \leq \left(\frac{ea\overline{X}y}{1+ah\overline{X}} - \frac{eay^2}{1+ah\overline{X}}\right)dt + \ell_2 y d\mathcal{B}_2$$

Let $\overline{Y}$ be the solution of the equation,

$$\begin{cases} d\overline{Y} = \left(\frac{ea\overline{X}\overline{Y}}{1+ah\overline{X}} - \frac{ea\overline{Y}^2}{1+ah\overline{X}}\right)dt + \ell_2\overline{Y}d\mathcal{B}_2 \\ \overline{Y} = \overline{Y}_0 \end{cases}. \quad (18)$$

Then (18) can be written as,

$$d\overline{Y} = \left(\frac{ea\overline{X}}{1+ah\overline{X}} - \frac{ea}{1+ah\overline{X}}\overline{Y}\right)\overline{Y}dt + \ell_2\overline{Y}d\mathcal{B}_2. \quad (19)$$

By the same way the solution of (19) is,

$$\overline{Y} = \frac{e^{\left(\int_0^t \frac{ea\overline{X}(s)}{1+ah\overline{X}(s)}ds - \frac{1}{2}(\ell_2)^2\right)t+\ell_2\mathcal{B}_2}}{\frac{1}{\overline{Y}_0} + ea\int_0^t \frac{1}{1+ah\overline{X}(s)} e^{\left(\int_0^s \frac{ea\overline{X}(l)}{1+ah\overline{X}(l)}dl - \frac{1}{2}(\ell_2)^2\right)s+\ell_2\mathcal{B}_2(s)}\,ds}.$$

Then we have, $y \leq \overline{Y}$, for all $t \in [0, t_E]$.

On the other hand, for the prey, from the positivity of x we have,

$$dx \geq \left(rx\left(1-\frac{x}{k}\right) - \varsigma\overline{Y} - \gamma x\right)dt + \ell_1 x d\mathcal{B}_1.$$

Where $\varsigma\overline{Y}$ is the largest functional response of the system (2), That means,

$$dx \geq \left(rx - \gamma x - \frac{r}{k}x^2 - \varsigma\overline{Y}\right)dt + \ell_1 x d\mathcal{B}_1, \quad (20)$$

Let $\varphi = r - \gamma$, then (20) can be written as,

$$dx \geq \left(\varphi x - \frac{r}{k}x^2 - \varsigma\overline{Y}\right)dt + \ell_1 x d\mathcal{B}_1$$

Let $\underline{X}$ be the solution of the equation,

$$\begin{cases} d\underline{X} = \left(\varphi\underline{X} - \frac{r}{k}\underline{X}^2 - \varsigma\overline{Y}\right)dt + \ell_1\underline{X}d\mathcal{B}_1 \\ \underline{X} = \underline{X}_0 \end{cases}. \quad (21)$$

The solution of (21) can be expressed by,

$$\underline{X} = \frac{e^{\left(\varphi - \frac{1}{2}(\ell_1)^2\right)t - \int_0^t \varsigma\overline{Y}(s)ds + \ell_1\mathcal{B}_1}}{\frac{1}{\underline{X}_0} + \frac{r}{k}\int_0^t e^{\left(\varphi - \frac{1}{2}(\ell_1)^2\right)s - \int_0^s \varsigma\overline{Y}(l)dl + \ell_1\mathcal{B}_1(s)}\,ds}.$$

Then we have, $x \geq \underline{X}$, for all $t \in [0, t_E]$.

The above results can by expressed as, $\underline{X} \leq x \leq \overline{X}$ and $\underline{Y} \leq y \leq \overline{Y}$.

Clearly $\overline{X}, \underline{X}, \overline{Y}$ and $\underline{Y}$ are exist for all time which in turn indicate $t_E = \infty$ and (x, y) is global solution.

The long-time behavior

From an ecological standpoint, species extinction is detrimental, whereas mutual coexistence is favorable. This section dedicates itself to delineating the conditions for the persistence and potential extinction of species within system (2).

Definition 2 [22]: Let, the following stochastic deferential equation,

$$dX(t) = a(X(t), t)dt + b(X(t), t)d\mathcal{B}(t),$$

represents a biological interaction, then the solution of it can be called,

- persistent in mean if,

$$\lim_{t\to\infty} inf \frac{1}{t}\int_0^t X(s)\, ds > 0.$$

- extinct if,

$$\lim_{t\to\infty} lnX(t) = 0.$$

Theorem 3: *The population of prey in system (2) will extinct if* $r < \frac{(\ell_1)^2}{2}$.

Proof: form theorem (2) It is clear that,

$$\lim_{t\to\infty} \underline{X} = 0.$$

If,

$$r < \frac{(\ell_1)^2}{2}. \tag{22}$$

Also,

$$\lim_{t\to\infty} \overline{X} = 0.$$

If,

$$r < \frac{(\ell_1)^2}{2} + \gamma. \tag{23}$$

It is clear that fulfilling condition (22) ensures the occurrence of condition (23).

This implies that if $r < \frac{(\ell_1)^2}{2}$, the prey population of system (2) will become extinct, which means,

$$\lim_{t\to\infty} x = 0.$$

Theorem 4: *If the prey population is extinct, then the predator population will become extinct without any condition.*

Proof: From the given hypothesis, it is concluded that,

$$\lim_{t\to\infty} x = 0.$$

Thus, the predator equation of system (2) can be written by,

$$dlny = \left(-eay - u - \delta - 0.5{\ell_2}^2\right)dt + \ell_2 d\mathcal{B}_2,$$

Now consider,

$$f(y) = -eay - u - \delta - 0.5{\ell_2}^2.$$

Where,

$$f'(y) = -ea < 0$$

Which indicates that $f(y)$ is decreasing in $[0, \infty)$, and the maximum value of it occurred when $y = 0$, that means,

$$dlny \leq \left(-u - \delta - 0.5{\ell_2}^2\right)dt + \ell_2 d\mathcal{B}_2, \tag{24}$$

by taking the integration to both sides of (24)

$$lny \leq lny_0 + \int_0^t \left(-u - \delta - 0.5{\ell_2}^2\right) ds + \ell_2 \int_0^t d\mathcal{B}_2(s),$$

$$lny \leq lny_0 + \left(-u - \delta - 0.5{\ell_2}^2\right)t + \ell_2\mathcal{B}_2. \tag{25}$$

By dividing both sides of (25) by t, and taking the limit when $t \to \infty$,

$$\lim_{t\to\infty} \frac{lny}{t} \leq -u - \delta - 0.5{\ell_2}^2 < 0.$$

From the above inequality, it is straightforward that the predator population approaches to zero as t approaches infinity,

$$\lim_{t\to\infty} y = 0.$$

Theorem 5: *When the prey is extinct the predator does not have a limit in mean.*

Proof: consider the predator equation in system (2),

$$dy = \left(\frac{eaxy}{1 + ahx} - \frac{eay^2}{1 + ahx} - uy - \delta y\right) dt + \ell_2 y d\mathcal{B}_2. \tag{26}$$

Eq. (26) can be written as,

$$dy = \left(\frac{eaxy}{1 + ahx} + \frac{ea^2hxy^2}{1 + ahx} - eay^2 - uy - \delta y\right) dt + \ell_2 y d\mathcal{B}_2.$$

Suppose that $V_1 = lny$, then by using Ito formula,

$$dlny = \left(\frac{eax}{1 + ahx} + \frac{ea^2hxy}{1 + ahx} - eay - u - \delta\right) dt + \ell_2 d\mathcal{B}_2 - \frac{{\ell_2}^2}{2} dt,$$

$$dlny = \left(\frac{eax}{1 + ahx} + \frac{ea^2hxy}{1 + ahx} - eay - u - \delta - \frac{{\ell_2}^2}{2}\right) dt + \ell_2 d\mathcal{B}_2.$$

For arbitrary $\varepsilon_1 > 0$ there exist t_0 and Ω_{ε_1} such that $\mathcal{P}\left(\Omega_{\varepsilon_1}\right) \geq 1 - \varepsilon_1$, and $\frac{eax}{1+ahx} \leq \varepsilon_1$, for $t \geq t_0$, and $\omega \in \Omega_{\varepsilon_1}$,

$$dlny \leq \left(\varepsilon_1 + \varepsilon_1 ahy - eay - u - \delta - \frac{{\ell_2}^2}{2}\right) dt + \ell_2 d\mathcal{B}_2.$$

By integrating both sides of the above inequality,

$$lny \leq lny_0 + \left(\varepsilon_1 - u - \delta - \frac{{\ell_2}^2}{2}\right) t + (\varepsilon_1 ah - ea) \int_0^t y\, ds + \ell_2\mathcal{B}_2.$$

Dividing both sides by t,

$$\frac{lny}{t} \leq \frac{lny_0}{t} + \left(\varepsilon_1 - u - \delta - \frac{{\ell_2}^2}{2}\right) + \frac{(\varepsilon_1 ah - ea)}{t} \int_0^t y\, ds + \frac{\ell_2\mathcal{B}_2}{t}.$$

Setting $t \to \infty$, then it can be concluded,

$$-(\varepsilon_1 ah - ea) \lim_{t\to\infty} inf \frac{1}{t} \int_0^t y\, ds \leq \left(\varepsilon_1 - u - \delta - \frac{{\ell_2}^2}{2}\right),$$

$$\lim_{t\to\infty} inf \frac{1}{t} \int_0^t y\, ds \leq \frac{\varepsilon_1 - u - \delta - \frac{{\ell_2}^2}{2}}{-(\varepsilon_1 ah - ea)}.$$

Since ε_1 is arbitrary then,

$$\lim_{t\to\infty} inf \frac{1}{t} \int_0^t y\, ds \leq \frac{-u - \delta - \frac{{\ell_2}^2}{2}}{ea} < 0.$$

That means the predator does not have a limit in mean, since the predator is depends entirely on the prey population.

Corollary 6: *If* $u+\delta+\frac{\ell_2{}^2}{2}>\varepsilon_1$*, then the predator may be extinct even the prey persistent.*
Consider the following inequality,

$$dlny \le \left(\varepsilon_1+\varepsilon_1 ahy-eay-u-\delta-\frac{\ell_2{}^2}{2}\right)dt+\ell_2 yd\mathcal{B}_2. \tag{27}$$

By integrating both sides of (27),

$$lny-lny_0 \le \left(\varepsilon_1-u-\delta-\frac{\ell_2{}^2}{2}\right)t+\varepsilon_1 ah\int_0^t F(y)ds-ea\int_0^t F(y)ds+\ell_2\mathcal{B}_2. \tag{28}$$

Dividing both sides of (28) by t and taking the limit when $t\to\infty$,

$$\lim_{t\to\infty}\frac{lny}{t}\le\varepsilon_1-u-\delta-\frac{\ell_2{}^2}{2}<0.$$

Now, if,

$$u+\delta+\frac{\ell_2{}^2}{2}>\varepsilon_1. \tag{29}$$

Then,

$$\lim_{t\to\infty}lny=0.$$

Theorem 7: *If* $r>\gamma+0.5\ell_1{}^2+\Upsilon$*, the prey will persistent regardless the predator density.*
Proof: by applying Ito formula to the prey equation in system (2) when $V_2=lnx$,

$$dlnx=\left[r\left(1-\frac{x}{k}\right)-\frac{ay}{1+ahx}-\gamma-0.5\ell_1{}^2\right]dt+\ell_1 d\mathcal{B}_1 \tag{30}$$

By solving Eq. (29),

$$lnx=lnx_0+\left(r-\gamma-0.5\ell_1{}^2\right)t-\frac{r}{k}\int_0^t x\,ds-\int_0^t\frac{ay}{1+ahx}ds+\ell_1\mathcal{B}_1 \tag{31}$$

Dividing (30) by t and taking the limit when $t\to\infty$,

$$\lim_{t\to\infty}\frac{1}{t}\left[\frac{r}{k}\int_0^t x\,ds+\int_0^t\frac{ay}{1+ahx}ds\right]=r-\gamma-0.5\ell_1{}^2$$

Which implies that,

$$\lim_{t\to\infty}inf\frac{1}{t}\int_0^t x\,ds\ge\frac{k}{r}\left[r-\gamma-0.5\ell_1{}^2-\Upsilon\right]>0$$

Then the prey will persistent, if,

$$r>\gamma+0.5\ell_1{}^2+\Upsilon, \tag{32}$$

where $\Upsilon=\lim_{t\to\infty}\frac{1}{t}\int_0^t\frac{ay}{1+ahx}ds$.

Numerical simulations

To establish the validity of the theoretical findings, a simulation of model (1) was conducted using MATLAB, focusing on time series and phase portraits by employing the Euler technique. This technique was employed to model (1), which was expressed in the subsequent discrete form:

$$x_{k+1}=x_k+x_k\left(r\left(1-\frac{x_k}{k}\right)-\frac{ay_k}{1+ahx_k}-j(1-e^{-y_k})\right)\Delta t+\ell_1 x_k\sqrt{\Delta t}\mathscr{b}_k^x, \tag{33}$$

$$y_{k+1} = y_k + y_k\left(\frac{eax_k}{1+ahx_k} - \frac{eay_k}{1+ahx_k} - u - m\big(1+cos(x_k)\big)\right)\Delta t + \ell_2 y_k \sqrt{\Delta t}\,\mathscr{b}_k^y.$$

Where $\mathscr{b}_k^x$ and $\mathscr{b}_k^y$ are the Gaussian random variables, and the dataset of parameters are given in the following table.

Table 2. *Dataset of selected parameters for deterministic system (3.2).*

Name	**Parameter**	**Value**
Growth rate of the prey population	r	3.1
Rate of predation	a	2.3
Decay rate of the predator population	u	0.25
Half-saturation constant	h	0.51
Conversion efficiency	e	0.35
Carrying capacity of prey	k	4
Proportion of predators capable of enhancing their predation skills	j	0.7
Proportion of prey that have the opportunity to escape or hide from predators	m	0.5

Also ℓ_1 and ℓ_2 will illustrated in each figure that required their value.

In numerical simulations performed on Model 1, the existence and uniqueness of the global positive solution were explored as shown in Fig. 1 and 2 which depend on 11 distinct initial conditions. Where in Fig. 1 the phase portrait provides visual confirmation of the theoretical results regarding the existence of a positive solution. The trajectories shown, which start from a set of initial states, coalesce into a specific path as time progresses. This convergence largely points to the inherent behavior of the model, supporting the existence of a positive solution which ecologically means that predators and prey co-exist in the environment. Moreover, despite the random fluctuations embedded in the system due to random fluctuations, the consistency with which the trajectories move toward a common region supports the uniqueness of the solution.

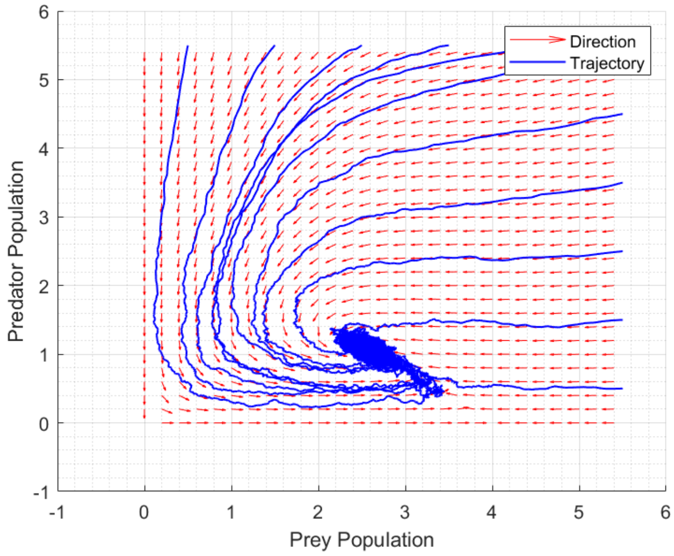

Fig. 1 The phase portrait of model (1) with 11 distinct initial conditions.

Complementing the phase portrait, the time series as shown in Fig. 2 shows the temporal stability of the model solutions. It shows how temporal evolutions of both prey and predators, regardless of initial conditions, show convergence toward a unique trajectory. The prey (blue) and predator (red) populations, after initial oscillations, display a tendency to reach a steady-state dynamic, reinforcing the presence of a unique positive solution. The uniqueness is further highlighted by the similarity in population dynamics across all stochastic realizations of the model, implying that a single positive solution dominates the long-term behavior of the system.

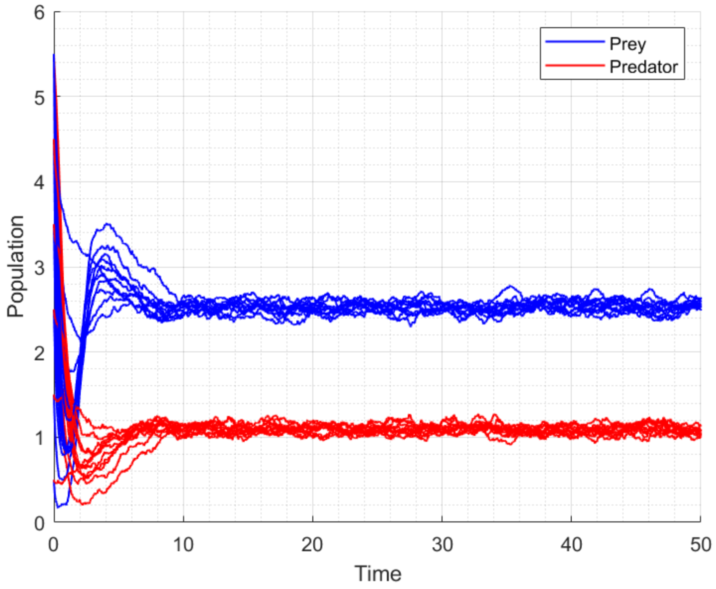

Fig. 2 The time series of model (1) with 11 distinct initial conditions.

Collectively, these simulations do not merely suggest the existence and uniqueness of solutions but affirm the stability of the stochastic prey-predator model numerically. This solution, consistent across various initial conditions and resilient to stochastic disturbances, validates our theoretical analysis and offers a concrete representation of the prey-predator dynamics as modeled.

According to the long-time behavior Theorem (3) predicts that the prey population will go extinct if the growth rate is less than half of the square of the noise intensity coefficient. where the dynamics of the prey population in a stochastic prey-predator model when condition (22) is violated is illustrated in Fig. 3.

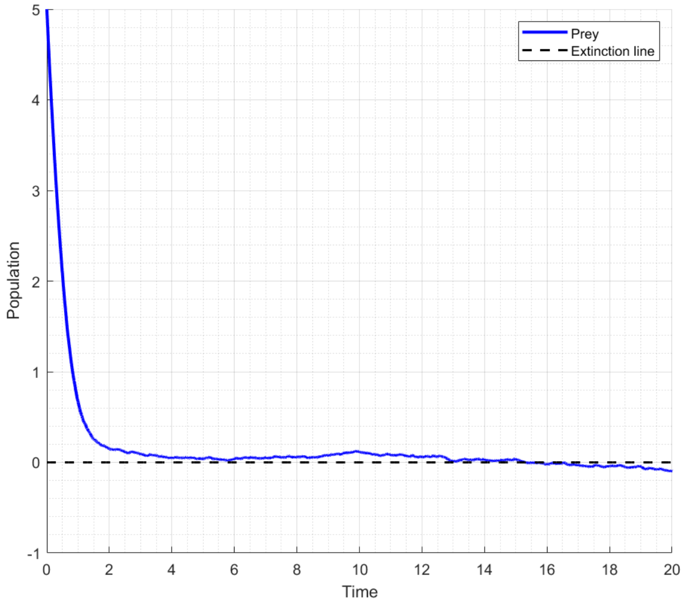

Fig. 3 The prey population when condition (22) is violated.

The trajectory of the prey population, represented by the blue line, shows a sharp decline towards the black dashed line, which symbolizes the extinction threshold. The convergence of the prey population to this extinction line without recovery illustrates that the prey population indeed goes extinct as per the theoretical prediction. This visual result supports Theorem (3), which states that under certain conditions the prey population cannot sustain itself and will inevitably decline to extinction. It is notable that the population declines to zero without subsequent fluctuations, which reinforces the certainty of extinction once the critical threshold for the growth rate is crossed.

Corresponding to Theorem (4), the predator population follows a similar trajectory, plummeting towards the extinction line shortly after the prey population does, as shown in Fig. 4.

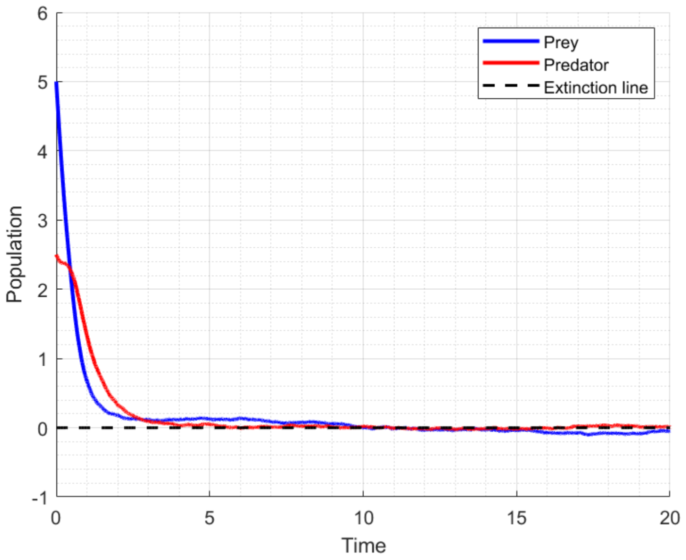

Fig. 4 The extinction of prey and predator populations when condition (22) is violated.

The immediate and correlated decline of the predator population, in this case, underscores the dire consequences of prey extinction, which in turn affirms the Theorem (4) that claims the predator population is reliant on the prey population for sustenance and, therefore, will become extinct if the prey population is extinguished.

In reference to Theorem (5), which states that when the prey is extinct, the predator does not have a limit in mean, Fig. 4 also suggests a rapid convergence to zero for the predator population after the prey's extinction. However, due to the stochastic nature of the model, it might expect some fluctuations around the extinction threshold due to the random perturbations inherent in such systems. where the predator fluctuations can be absent around the extinction line due to a multitude of factors, including a strong dependency of predators on prey or high rates of predator death without prey.

Numerical simulations were also applied to the model (1), where condition (29) occurs. This simulation presented a wonderful aspect related to the long-term behavior of the stochastic prey and predator models, as shown in Fig. 5.

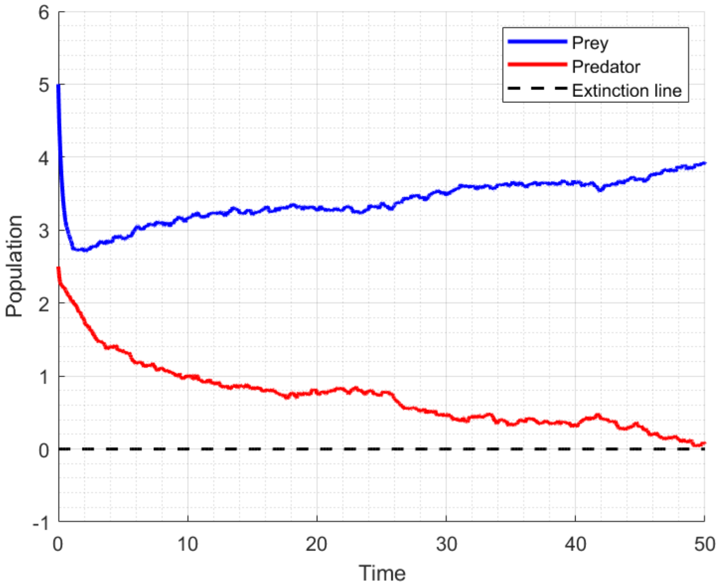

Fig. 5 The populations of prey and predator in model (1) when condition (29) occurs.

In Fig. 5, it was observed that the prey curve reached a state of permanence. Despite this, predator numbers decline sharply, eventually converging on the extinction threshold. This numerical result supports Corollary (6), suggesting that when the combined effects of predator decay and noise overcome the efficiency of converting prey into predator biomass, it leads to the extinction of predators, despite the presence of a large number of preys.

Further simulations have been conducted to validate Theorem (7), which proposes that prey permanence is assured under specific growth rate conditions, irrespective of predator density. Two scenarios were considered, with initial predator densities set at 2.5 and 7.5, while the initial prey density was maintained at 5 for both as illustrated in Fig. 6 and 7.

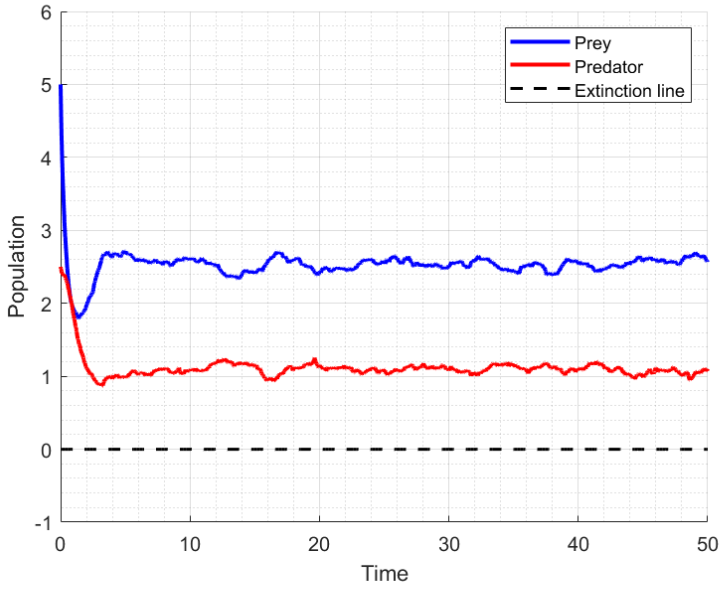

Fig. 6 The populations in model (1) when the initial condition is (5,2.5).

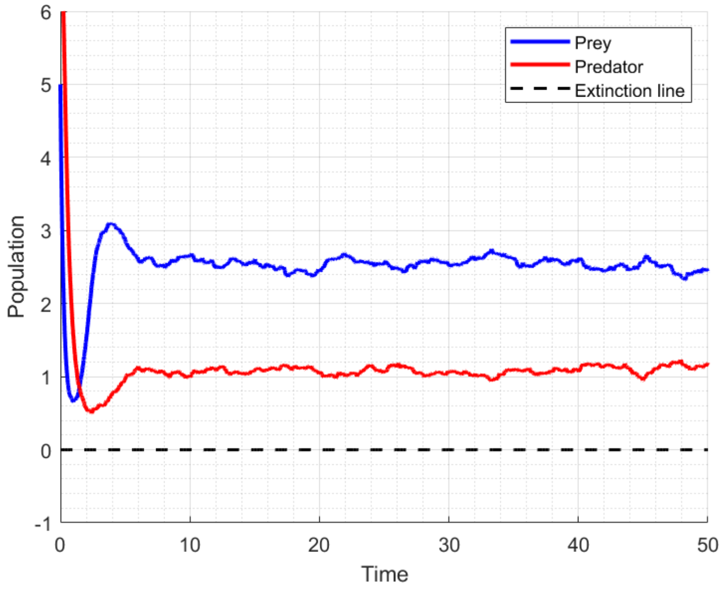

Fig. 7 The populations in model (1) when the initial condition is (5,7.5).

The resulting time series, presented in the previous figures, confirm that the prey population achieves permanence, as evidenced by the blue curves. This outcome occurs despite the differing predator densities, which eventually stabilize at distinct levels shown by the red curves. The consistency of the prey population's behavior in both simulations supports the theorem's prediction that, above a certain threshold for r, the prey population's persistence is unaffected by the initial or consequent predator density. These findings reinforce the concept that certain intrinsic growth rates can ensure species permanence within ecological models, even in the presence of predation and stochastic events.

Conclusion

In conclusion this study, we rigorously examined a stochastic prey-predator model under the predation skill augmentation for predator and hide-and-escape Effect for prey effects, confirming the existence and uniqueness of global positive solutions. The numerical simulations aligned seamlessly with the theoretical underpinnings, demonstrating the resilience of prey populations under varying predator densities and stochastic conditions. We also highlighted the critical thresholds of growth rates and predation efficiencies that dictate the permanence or extinction of species. Our findings provide insightful contributions to the ecological modeling field, offering predictive understandings of population dynamics under stochastic influences, with implications for ecological conservation and management strategies.

Declarations:

Competing interests: The authors declare no conflict of interest.

Authors' contributions: All authors contributed equally in this work.

Data availability: The datasets and the MATLAB code used during the current study available from the corresponding author on reasonable request.

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السلوك طويل الامد لنموذج الفريسة والمفترس العشوائي مع تاثير الهروب والاختفاء للفريسة وقابيلة زيادة مهارة الافتراس للمفترس

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البحث مستل من اطروحة دكتوراه الباحث الاول

معلومات البحث:
تأريخ الاستلام:
تاريخ التعديل :
تأريخ القبـــول:
تاريخ الـــنشر:

الكلمات المفتاحية:
نموذج الفريسة والمفترس، الاستجابة الوظيفية، المعادلات التفاضلية التصادفية، التعايش، الانقراض

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الخلاصة:

في هذا البحث، تم اجراء تحقيق شامل لنموذج الفريسة والمفترس مع الاستجابة الوظيفية والعددية من نوع هولينج الثاني تحت تاثير مؤثر الاختفاء والهروب للفريسة وزيادة مهارة الافتراس للمتفرس. في البداية تم توضيح اهمية التاثيرات الجديدة وكيفية نمذجتها رياضيات ودمجها في النموذج الاساسي. تم اجراء تحليل شامل على النومج المقترح. من الناحية النظرية فقد تم اثبات ان للنموذج حلا وحيدا في المنطقة الموجبة، وهذا الحل هو وحيد وعالمي. ايضا تم دراسة ظروف التعايش والانقراض للانواع في النموذج. اما من الناحية العددية فقد تم محاكاة النموذج باستخدام برنامج الماتلاب، حيث تم التحقق من صحة النتائج النظرية.