

Error Analysis of the Classical Artificial Diffusion Weak Galerkin Finite Element Method for the steady state- convection diffusion-reaction Equation in 2-D

Ibtihal Ahmed Abd^{1*} and Hashim . A . Kashkool¹

1- Department of mathematics, college of education for pure sciences, university of Basrah, Basrah, Iraq



This work is licensed under a [Creative Commons Attribution 4.0 International License](https://creativecommons.org/licenses/by/4.0/)

<https://doi.org/10.54153/sjpas.2024.v6i2.893>

Article Information

Received: 14/04/2024

Revised: 21/05/2024

Accepted: 28/05/2024

Published: 30/06/2024

Keywords:

Classical artificial diffusion, Convection Diffusion Reaction problem, Error analysis, stability.

Corresponding Author

E-mail:

ibtehalahmedalsady@gmail.com

Mobile: +9647728407704

Abstract

In this study, we modified the error in the weak Galerkin method when solving problems in which diffusion is the dominant convection ($\epsilon \leq \|\beta\|_{L^\infty(\Lambda)}h$) in two dimensions. This is done by adding the artificial diffusion term $(-\delta\Delta w, \text{ where } \delta=h-\epsilon)$. The finite element method for discrete functions using a weakly defined gradient operator is presented in this study. The concept of the weak discrete gradient is introduced, which plays an important role when using numerical methods to solve partial differential equations. The goal of this study is to enhance the accuracy and stability of the solutions by studying the ellipticity and stability properties of the method, which works to ensure that the numerical method retains the properties of the original equation while reducing the fluctuations occurring with the weak galerkin finite element method. Specific theories have been used to estimate the error in parameters H^1 -norm, and . Practical examples demonstrate how this method improves the handling of partial equations characterized by convection-dominated diffusion, enhancing its potential for advancing numerical simulations in engineering and physics.

Introduction:

This study presents a numerical method for approximating partial differential equations, based on a new understanding of differential operators and their approximations. To express this method, we analyze the Dirichlet problem for elliptic equations in second order, which includes determining a function with an unknown value. The function w depends on the variable z and satisfies the following:

$$-\epsilon\Delta w + w_\beta + w = f \quad \forall w \in \Lambda, \quad (1)$$

$$w = g \text{ on } \Gamma \quad (2)$$

Here, Λ is a bounded convex polygonal domain in \mathbb{R}^2 with boundary Γ and $\beta = (\beta_1, \beta_2)$ is a constant vector with $\|\beta\|_{L^\infty(\Lambda)} = 1$. The symbols ϵ and $w_\beta = \beta \cdot \nabla w$ denote a positive

constant and the derivative in the β -direction, respectively. The gradient operator is denoted by ∇ , and ∇w represents the gradient of the function $w = w(z_1, z_2)$ [1].

Such problems appear fundamental in describing various physical processes in science and engineering, such as in viscous media represented by the dispersion of pollutants, in semiconductors as electron flow, and over bird wings as airflow. The solution to problems (1) and (2) depends on the relationship between the diffusion parameter (ϵ) and the lattice length (h). Specifically, when ϵ drops below $\|\beta\|_{L^\infty(\Lambda)} h$, where $\|\beta\|_{L^\infty(\Lambda)} = 1$. However, caution must be taken when ϵ is less than h , as this can lead to oscillatory profiles that deviate significantly from the exact solution. This inherent limitation compromises the required accuracy and overall smoothness, making numerical solutions with conventional Galerkin estimation unsatisfactory. This requires developing a more accurate and consistent numerical estimation strategy[1].

Various stabilization techniques, such as SUPG, [2, 3], Upwinding techniques [4, 5], LPS [6, 7], and discontinuous Galerkin methods [8,9,10]. Streamline diffusion method [11,4], Several alternatives were compared in [12,13, and Have been proposed to avoid these oscillations. However, no completely satisfactory method has been found yet. For instance, the popular SUPG approach tends to generate too many peaks and valleys near the stratigraphic area.

Wang and Ye have introduced the weak Galerkin (WG) method, a new finite-element framework for numerically solving second-order elliptic equations. This approach, described in their recent works [14,15], incorporates weak functions and gradients, allowing degrees of freedom on both elements and edges. The WG approach permits the use of completely discontinuous functions as basic functions. This technique allows the use of versatile meshes and provides a simple and parameter-free method. It is intended to solve various partial differential equations, such as second-order elliptic problems, Stokes equations, biharmonic equations, and interface problems [16,17,18,19]. Given the computational challenges arising from accurately reproducing physical processes while at the same time ensuring the speed of the solution, it is necessary to find a balance between computational efficiency and physical consistency. This can be achieved by following numerical methods that achieve both goals. In the following paper, WGFEM was enhanced by improvements to the low-order Galerkin finite element method (WGFEM) in a way that enabled linear convective diffusion problems to be addressed. This is done by adding the diffusion term $-\delta\Delta w$, with $\delta=h-\epsilon$, in the problem statement. , then following the strong and weak Galerkin (WG) approach that solves the diffusion equation in the circumstance where $h > \epsilon$ or in situations where convection governs diffusion.

In this paper, we will use the standard notation of Sobolev space with it as the norm and their associated inner product, seminorms[14].

Weak Galerkin finite element method

The goal of this section is to design a weak finite element Galerkin scheme. Initially, most cases applied to the numerical solution of PDEs in science and engineering involve the finite

element method. In recent years, the researcher has presented a new and innovative finite element model called WGFE, which excels as a field-solving model (FM), which was presented in 2013 by J. Wang and X. Yes. WGFE generally works when convection is the dominant process compared to diffusion. It breaks the complex PDE into smaller parts by first finding the weak form of this equation. Having a polynomial function to approximate the placement of modules within each element is a proper measure. The original problem is solved by WGFEM using different approximation spaces for the solution and the gradient. This strategy is used in situations where the solutions might be very discontinuous or rapidly change. In the WG-FEM steps, unknowns are known both within the elements and on the element borders, and the weak gradient of basis functions can be solved element by element. Compared with traditional GFEMs, the WG methods are more appropriate for solving problems with discontinuous solutions or on complicated domains. Moreover, compared with the conventional discontinuous Galerkin (DG) methods, there is no need to choose “large enough” parameters in a stabilization term (even no penalty terms in some types of WG-FEM). The WG methods have compact formulations that can be applied near boundaries without special treatment, greatly increasing the robustness and accuracy of boundary condition implementation. Let us consider an Λ triangular partition referred to as Z_h whose set of components is composed of connected and closed polygons in R^2 , which satisfy some assumptions about how regular the elements are. Denote by h_T the diameter for every element $T \in Z_h$ and $h = \max_{T \in Z_h} h_T$ the mesh size for Z_h . For a given integer $j, l \geq 1$, let $\rho^h(j, l)$ be the weak Galerkin finite element space associated with Z_h

$$\rho^h(j, l) = \{\rho = \{\rho_0, \rho_b\}: \rho_0|_T \in \mathbb{P}_j(T), \rho_b|_{\partial T} \in \mathbb{P}_l(\partial T), T \in Z_h\},$$

and

$$\rho_0^h(j, l) = \{\rho = \{\rho_0, \rho_b\}: \rho_0 \in \rho^h(j, l), \rho_b = 0 \text{ on } \partial T \cap \partial \Lambda\},$$

where $\mathbb{P}_j, \mathbb{P}_l$ is the space of polynomials of total degree j, l or less. For any $\rho = \{\rho_0, \rho_b\}$, the discrete weak gradient $\nabla_r \rho \in [\mathbb{P}_r(T)]^2$ is defined on T as the unique polynomial satisfying

$$\int_T \nabla_r \rho \cdot q dT = - \int_T \rho_0 \nabla \cdot q dT + \int_{\partial T} \rho_b q \cdot \mathbf{n} ds, \forall q \in [\mathbb{P}_r(T)]^2$$

where \mathbf{n} is the unit outward normal vector to ∂T . For any $\rho = \{\rho_0, \rho_b\}$, we define the weak directional derivative $\beta \cdot \nabla_r \rho \in \mathbb{P}_r(T)$ related to $\beta \cdot \nabla \rho$ on T as the unique polynomial satisfying

$$\int_T \beta \cdot \nabla_r \rho u dT = - \int_T \rho_0 \nabla \cdot (\beta u) dT + \int_{\partial T} \rho_b \beta \cdot \mathbf{n} u ds \quad \forall u \in \mathbb{P}_r(T).$$

We introduce four global projections Q_0, Q_b, Q_h , and \mathbb{Q}_h . They are element-wise defined L^2 projections detailed as follows: For each element $T \in Z_h$, $Q_0: L^2(T) \rightarrow \mathbb{P}_r(T)$ and $Q_b: L^2(\partial T) \rightarrow \mathbb{P}_r(\partial T)$ are the L^2 projections onto the associated local polynomial spaces, $\mathbb{Q}_h: [L^2(T)]^d \rightarrow [\mathbb{P}_{r-1}(T)]^d$ is the L^2 projection onto the local weak gradient space. Finally, we define a projection operator $Q_h w = \{Q_0 w, Q_b w\} \in \rho^h(j, l)$ for the true solution w [17]

$$\nabla_r(Q_h u) = R_h(\nabla u), \quad \forall u \in H^1(T). \quad (3)$$

also define a projection π_h such that $\pi_h \mathbf{q} \in H(\text{div}, \Omega)$, and on each $T \in \mathcal{Z}_h$, one has $\pi_h \mathbf{q} \in V(T, r = j + 1)$ and the following identity

$$(\nabla \cdot \mathbf{q}, v_0)_T = (\nabla \cdot \pi_h \mathbf{q}, v_0)_T, \quad \forall v_0 \in P_j(T^0). \quad (4)$$

The WGFE method is expressed as follows: Find $w_h = \{w_0^h, w_b^h\} \in \rho_0^h(j, l)$ such that

$$(\epsilon \nabla_r w_h, \nabla_r \rho) + (\beta \cdot \nabla_r w_h, \rho_0) + (w_h, \rho_0) = (f, \rho_0), \quad \forall \rho = \{\rho_0, \rho_b\} \in \rho_0^h \quad (5)$$

where

$$\begin{aligned} (\epsilon \nabla_r w_h, \nabla_r \rho) &= \int_{\Lambda} \epsilon \nabla_r w_h \cdot \nabla_r \rho d\Lambda, \\ (\beta \cdot \nabla_r w_h, \rho_0) &= \int_{\Lambda} \beta \cdot \nabla_r w_h \rho_0 d\Lambda, \\ (w_0, \rho_0) &= \int_{\Lambda} w_0 \rho_0 d\Lambda. \end{aligned}$$

The classical artificial diffusion Weak Galerkin Finite Element Method

Numerical simulations have difficulties in accurately representing steep gradients and rapid changes in solution when the mesh size exceeds the diffusion limit. However, it becomes more complex in the convective-dominated model, mainly because such transfer of properties is a significant occurrence in most of the cases. Therefore, the numerical solution may face problems such as instability, oscillations, or an inability to represent the physical behavior of the system accurately. Inaccuracies and errors can arise due to insufficient resolution to capture small-scale features and rapid changes in the solution. To address these challenges, digital technology has introduced artificial diffusion (AD), where a controlled amount of artificial or numerical diffusion is introduced to stabilize the numerical solution. This additional spread helps facilitate the solution, preventing fluctuations or instability that may arise due to inaccuracy [21,22]. The WGFE method, as shown in equation (5), tends to produce oscillatory solutions when $\epsilon < \|\beta\|h$ and the exact solution is not smooth enough. Therefore, the WGFE method may produce suboptimal results. To address these challenges, we will analyze the WGFE method by integrating the classical artificial diffusion technique to solve the linear thermal diffusion problem. To effectively mitigate the problems associated with the weak Galerkin method (Equation 5) when $\epsilon < \|\beta\|h$, where $\|\beta\| = 1$, leads to $\epsilon < h$ we avoid such cases altogether. This can be achieved in two ways: by reducing h to the point where $\epsilon > h$, which may be impractical when ϵ is very small, or by addressing the problem through a modified approach by adding $-\delta \Delta w$, where $\delta = h - \epsilon$ to the original equation then follow the approach WGFE method. This is the basic concept behind the classical artificial diffusion weak Galerkin finite element method (CADWGFEM). The CADWGFEM is a finite element method designed to determine $w_h = \{w_h^0, w_h^b\} \in \rho^h$ by using equations (1) and (2). The primary goal of CADWGFEM is to satisfy the condition $w^b = Q^b g$ on Γ . The expression for the CADWGF approximation is as follows:

Find $w_h = \{w_0^h, w_b^h\} \in \rho_0^h$ so that:

$$a_{CAD}(w_h, \rho) = (f, \rho_0), \quad \forall \rho = \{\rho_0, \rho_b\} \in \rho_0^h. \quad (6)$$

And where to

$$a_{CAD}(w_h, \rho) = (h\nabla_r w_h, \nabla_r \rho) + (\beta \cdot \nabla_r w_h, \rho_0) + (w_h, \rho_0) \quad (7)$$

Existence and Uniqueness for CADWGFE Approximations.

Let ρ_h represent a CADWGFE approximation for the given problem described by equations (1) and (2), derived from (6) by using the WG finite element space $\rho^h(j, l)$. The objective of this section is to establish two fundamental properties of CADWGFE for linear convection-diffusion problems: the V-elliptic property and the stabilization property.

Lemma 1. Let $a_{CAD}(\cdot, \cdot)$ represent the bilinear form, as described in equation (7). There is a positive constant φ such that

$$a_{CAD}(\rho_h, \rho_h) \geq \varphi(\|\nabla_r \rho_h\|^2 + \|\rho_0\|^2)$$

Proof. Using $w = \rho$ in equation (7) yields

$$a_{CAD}(\rho_h, \rho_h) = (h\nabla_r \rho_h, \nabla_r \rho_h) + (\beta \cdot \nabla_r \rho_h, \rho_0) + (\rho_0, \rho_0), \quad (8)$$

We obtain the following by applying the Cauchy-Schwarz inequality and using the fact $\|\beta\|_{L^\infty(\Lambda)} = 1$

$$\begin{aligned} |(\beta \cdot \nabla_r \rho_h, \rho_0)| &\leq \|\beta \cdot \nabla_r \rho_h\| \|\rho_0\| \\ &\leq \|\nabla_r \rho_h\| \|\rho_0\| \end{aligned}$$

And

$$(\rho_0, \rho_0) \leq \|\rho_0\|^2 \quad (10)$$

Substituting (9) and (10) into (8) gives the following result:

$$a_{CAD}(\rho_h, \rho_h) \geq h\|\nabla_r \rho_h\|^2 - \|\nabla_r \rho_h\| \|\rho_0\| + \|\rho_0\|^2$$

Now, let's apply the ε -Young's inequality, and we get:

$$\|\nabla_r \rho_h\| \|\rho_0\| \leq \frac{K}{2} \|\nabla_r \rho_h\|^2 + \frac{1}{2K} \|\rho_0\|^2$$

Substitute it for the original inequality:

$$a_{CAD}(\rho_h, \rho_h) \geq h\|\nabla_r \rho_h\|^2 + \left(\frac{K}{2} \|\nabla_r \rho_h\|^2 + \frac{1}{2K} \|\rho_0\|^2\right) + \|\rho_0\|^2$$

Combination of similar terms:

$$a_{CAD}(\rho_h, \rho_h) \geq \left(h + \frac{K}{2}\right) \|\nabla_r \rho_h\|^2 + \left(1 + \frac{1}{2K}\right) \|\rho_0\|^2$$

$$a_{CAD}(\rho_h, \rho_h) \geq \varphi(\|\nabla_r \rho_h\|^2 + \|\rho_0\|^2).$$

where $\varphi = \min\left\{h + \frac{K}{2}, \left(1 + \frac{1}{2K}\right)\right\}$

Lemma 2. (Stability) Suppose that Lemma 1 holds and the solution of equation (7) fulfills the condition that there exists a constant $\varphi > 0$, such that

$$(\|\nabla_r w_h\|^2 + \|w_0\|^2) \leq \frac{1}{2\varphi\mu} \|f\|^2 + \frac{\mu}{2\varphi} \|w_h\|^2$$

Proof. If we choose $\rho = w_h$ in equation (6), we get

$$a_{CAD}(w_h, w_h) = (f, w_h) \quad (11)$$

From Lemma 3 we have the following;

$$a_{CAD}(w_h, w_h) \geq \varphi(\|\nabla_r w_h\|^2 + \|w_0\|^2) \quad (12)$$

Using Cauchy-Schwarz and Young's inequalities, the first term on the right-hand side can be calculated.

$$\begin{aligned} (f, w_h) &\leq \frac{1}{\sqrt{\mu}} \|f\| \sqrt{\mu} \|w_h\| \\ &\leq \frac{1}{2\mu} \|f\|^2 + \frac{\mu}{2} \|w_h\|^2 \end{aligned} \quad (13)$$

Substituting Eq. (12) and (13) in Eq. (11), we get;

$$\varphi(\|\nabla_r w_h\|^2 + \|w_0\|^2) \leq \frac{1}{2\mu} \|f\|^2 + \frac{\mu}{2} \|w_h\|^2 \quad (14)$$

We divide by the value of φ in the equation (14), the result is as follows

$$(\|\nabla_r w_h\|^2 + \|w_0\|^2) \leq \frac{1}{2\varphi\mu} \|f\|^2 + \frac{\mu}{2\varphi} \|w_h\|^2 \quad (15)$$

As shown in Equation (6), we have demonstrated the stability of the CADWGFE method. This means that based on the given problem data, the appropriate criterion for the solution can be estimated.

Error Analysis

The purpose of this section is to provide an error estimate for CADWGFE (Equation 6). Our approach is consistent with traditional error analysis methodology and involves checking the discrepancy between the CAWGFE approximation w_h and a specific interpolation/projection of the exact solution by the error equation. We start the derivation with an error equation for the CADWGFE approximation, w_h and the projection L^2 to the exact solution w in the weak finite element space $\rho_h(j, l)$. Let $v = \{v_0, v_b\} \in \rho_h^0(j, l)$ be a random test function.

The CADWGFEM in equation (6) is consistent in the sense that it is satisfied by the exact solution w , i.e.

$$a_{CAD}(w, v_0) = (f, v_0), \quad \forall v_0 \in \rho_h^0(j, l) \quad (16)$$

subtracting Equ.(16) from Equ. (6) we have the following equation

$$a_{CAD}(\omega - \omega_h, v) = 0, \forall v \in \rho_h^0(j, l) \quad (17)$$

putting $w - w_h = w - Q_h w - (w_h - Q_h w)$ in Equ. (17), we have

$$a_{CAD}(w - Q_h w, v_0) - a_{CAD}(w_h - Q_h w, v_0) = 0 \quad (18)$$

and using the Equ. (3) and Equ.(4) for Equ.(18), we getting

$$a_{CAD}(w_h - Q_h w, v_0) = (\pi_h h \nabla_r w - h R_h(\nabla_r w), \nabla_r v) - (\pi_h w - Q_0 w, \nabla_r(\beta v)) + (w - Q_0 w, v_0). \quad (19)$$

The Equ. (19) shall be called the error equation for the CADWGF method.

1. An Estimate error in H^1 -Norm

Let's start with the following term. It gives an error estimate of the difference between the CADWGF approximation w_h and the L^2 projection of the exact solution of the original problem.

Lemma 3. [14] For $w \in \mathbb{H}^{1+s}(\Lambda)$ with $s > 0$, we get $\|\pi_h(\epsilon \nabla w) - \epsilon Q_h(\nabla w)\| \leq Ch^s \|w\|_{1+s}$.

Lemma 4. [14]: For $w \in \mathbb{H}^{1+s}(\Lambda)$ with $s > 0$, we have $\|\pi_h w - Q_h w\| \leq Ch^{s+1} \|w\|_{1+s}$

Lemma 5. For $w \in \mathbb{H}^{1+s}(\Lambda)$ with $s > 0$, we have $\|\pi_h(h \nabla w) - h \nabla(Q_h w)\| \leq Ch^{s+1} \|w\|_{1+s}$

Proof. From lemma 3 we, get

$$\begin{aligned} \|\pi_h(h \nabla w) - h Q_h(\nabla w)\| &\leq h \|\pi_h(\nabla w) - Q_h(\nabla w)\| \\ &\leq h(Ch^s \|w\|_{1+s}) \\ &\leq Ch^{s+1} \|w\|_{1+s} \end{aligned}$$

Lemma 6. let $\omega_h \in \rho^h(j, l)$ be a CADWGF solution of the problem (1,2) arising from (6). Assume that the exact solution $\omega \in H^{k+1}(\Lambda)$. Denote by $e_h = \omega_h - Q_h \omega$ the difference between the CADWGF approximation and the L^2 projection of the exact solution $\omega = (\omega_1, \omega_2)$. Then there exists a constant C_0 depends on h such that

$$\|e_0\|^2 + \|\nabla_r e\|^2 \leq C_0 h^{2k} \|\omega\|_{1+k}^2$$

Proof. Substituting v in Equ. (19) by $e_h := \omega_h - Q_h \omega$ and put $v = e_h$ we arrive at

$$a(w_h - Q_h w, e_h) = R_1 - R_2 + R_3$$

Where

$$\begin{aligned} R_1 &= (\pi_h(h \nabla_r \omega) - h R_h(\nabla \omega), \nabla_r e_h) \\ R_2 &= (\pi_h(\omega) - Q_0 \omega, \beta \nabla(e_h)) \\ R_3 &= (\omega - (Q_0 \omega), e_0) \end{aligned}$$

To estimate R_1 by Cauchy - Schwartz and Young's-inequality we obtain;

$$|R_1| \leq \frac{1}{2\emptyset} \|\pi_h(h\nabla_r \omega) - hR_h(\nabla \omega)\|^2 + \frac{\emptyset}{2} \|\nabla_r e\|^2$$

by lemma (5) We have;

$$|R_1| \leq Ch^{2k+2} \|\omega\|_{1+k}^2 + \frac{\emptyset}{2} \|\nabla_r e_h\|^2. \quad (21)$$

To find R_2 , by Cauchy - Schwartz and Young's-inequality and lemma(4), we obtain

$$\begin{aligned} |R_2| &\leq \emptyset \|\beta\|^2 \|\pi_h(\nabla_r \omega) - Q^0(\nabla_r \omega)\|^2 + \frac{1}{4\emptyset} \|e_0\|^2. \\ &\leq C\emptyset h^{2k} \|\omega\|_{1+k}^2 + \frac{1}{4\emptyset} \|e_0\|^2 \end{aligned}$$

To estimate R_3 by Cauchy - Schwartz and Young's-inequality and using lemma(4) we obtain;

$$\begin{aligned} |R_3| &\leq \emptyset \|\omega - (Q^0 \omega)\|^2 + \frac{1}{4\emptyset} \|e_0\|^2 \\ |R^3| &\leq C\emptyset h^{2k+2} \|\omega\|_{1+k}^2 + \frac{1}{4\emptyset} \|e_0\|^2 \end{aligned} \quad (23)$$

Substituting Eq. (21) and Eq. (23), into Eq. (20), and applying the lemma(1) to the left part we have ;

$$\begin{aligned} \alpha (\|e_0\|^2 + \|\nabla_r e\|^2) &\leq C \frac{\emptyset}{2} h^{2k+2} \|\omega\|_{1+k}^2 + C\emptyset h^{2k} \|\omega\|_{1+k}^2 + C\emptyset h^{2k+2} \|\omega\|_{1+k}^2 \\ &\quad + \frac{2}{4\emptyset} \|e_0\|^2 + \frac{\emptyset}{2} \|\nabla_r e\|^2 \end{aligned}$$

we obtain ;

$$\left(\alpha - \frac{\emptyset}{2}\right) (\|e_0\|^2 + \|\nabla_r e\|^2) \leq (C \frac{\emptyset}{2} h^2 + C\emptyset + C\emptyset h^2) h^{2k} \|\omega\|_{1+k}^2,$$

We dived by $(\alpha - \frac{\emptyset}{2})$, we have

$$(\|e_0\|^2 + \|\nabla_r e\|^2) \leq C_0 h^{2k} \|\omega\|_{1+k}^2,$$

Where $C_0 = \frac{(\frac{3}{2} C \emptyset h^2 + C\emptyset)}{(\alpha - \frac{\emptyset}{2})}$, and $\alpha > \frac{\emptyset}{2}$. (I prepared the calculation to be more accurate by correctly relying on)

Numerical Results:

In this numerical example, we aim to prove the theoretical ideas discussed previously by providing a concrete demonstration. To illustrate these concepts, we use test problems with well-established exact solutions as tools. Specifically, our focus is on demonstrating optimal convergence of the CADWGF scheme, even when chosen arbitrarily. The problem at hand relates to the linear convective diffusion equation (1), and we establish homogeneous dual bounds and initial assumptions. The main parameters include $\epsilon = 0.0001$, the field $\Lambda =$

$[0,1] \times [0,1]$, and the velocity vector $\beta = \left(\cos \left(\frac{\pi}{3} \right), \sin \left(\frac{\pi}{3} \right) \right)$. In addition, initial and boundary conditions, along with the source term $f(z)$ derived from the given data, are taken into account. To analyze the solution, we start by dividing the square domain $\Lambda = [0,1] \times [0,1]$ into $M \times M$ sub-quadrants. Next, a triangular mesh is created using a diagonal line with a negative oblique angle to divide each square element into two triangles. The spatial grid size is denoted by $h = 1/D$ with $D = 2,4,16,32,64$. Use function spaces $\mathbb{L}^2(\Lambda)$ and $\mathbb{H}^1(\Lambda)$. We calculate the errors $w - w_h$ based on the network size and present the results in Table 1, to validate our theoretical analysis. In addition, Figure 1 shows the numerical and exact solutions of the WG method with mesh size $h = \frac{1}{64}$, while Table 2 and Figure 2 show the results of the CADWGFEM method. In this specific example, we adopt the exact solution

$$w = \sin(\pi u) \sin(\pi v).$$

The CADWGFEM approximation (Equation 5) is represented as a matrix problem for each element, where the local stiffness matrix is crucial for implementing CADWGFEM on a computer. It is clear from Figure 1 that the exact solution does not agree with the approximate solution of the WGFE method, as the latter shows noticeable oscillations. Table I shows the results of the WGFE method. However, when comparing the exact solution with the modified method (CADWGFEM), it becomes clear that the method is almost indistinguishable from the exact solution, as shown in Figs. Hence, the numerical results in CADWGFEM show relatively lower errors than those of WGFEM when compared with $\mathbb{L}^2(\Lambda)$ and $\mathbb{H}^1(\Lambda)$. Standardize the values in the tables. Furthermore, it can be seen that the smaller the network, the more accurate the solution is, with fewer errors.

Table 1: \mathbb{H}^1 - error and \mathbb{L}^2 - error for WGFEM with, $\epsilon = 10^{-4}$.

h	\mathbb{L}^2 - Error	\mathbb{L}^2 -Order	\mathbb{H}^1 - Error	\mathbb{H}^1 -Order
2.5000e - 01	4.2423e - 01	0	6.3602e - 01	0
1.2500e - 01	1.0926e - 01	1.9071e + 00	3.3157e - 01	9.3 78e - 01
6.2500e - 02	2.7286e - 02	2.0516e + 00	1.6781e - 01	9.8150e - 01
3.1250e - 02	6.5814e - 03	2.0717e + 00	8.4535e - 02	9.8919e - 01
1.5625e - 02	1.4005e - 03	2.2224e + 00	4.3085e - 02	9.1236e - 01

Table 2: \mathbb{H}^1 - error and \mathbb{L}^2 - error for CADWGFEM with $\epsilon = 10^{-4}$.

h	\mathbb{L}^2 - Error	\mathbb{L}^2 -Order	\mathbb{H}^1 - Error	\mathbb{H}^1 -Order
2.5000e - 01	1.1682e - 02	0	3.1788e - 02	0
1.2500e - 01	3.0087e - 03	1.9571e + 00	1.6571e - 02	9.3978e - 01
6.2500e - 02	7.5135e - 04	2.0016e + 00	8.3868e - 03	9.8250e - 01
3.1250e - 02	1.8123e - 04	2.0517e + 00	4.2250e - 03	9.8918e - 01
1.5625e - 02	3.8565e - 05	2.2324e + 00	2.1534e - 03	9.7236e - 01

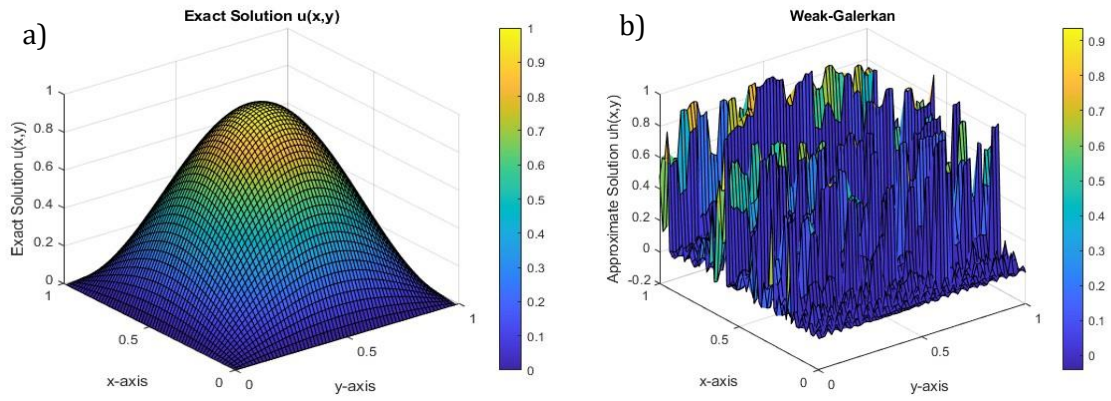


Fig. 1 a) exact solution with $\epsilon = 10^{-4}$ and b) wife numerical solution using $\epsilon = 10^{-4}$.

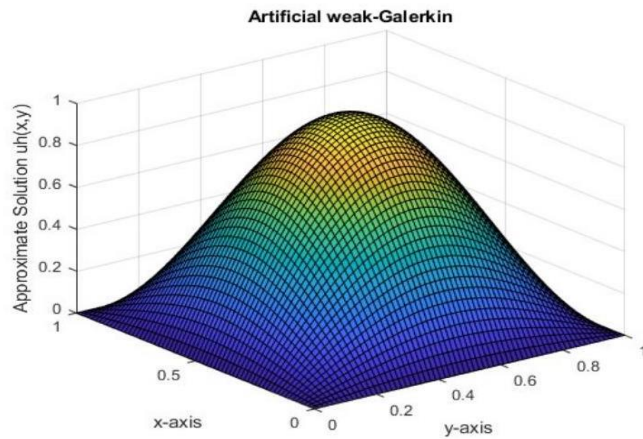


Fig. 2 numerical solution of CADWGFEM using $\epsilon = 10^{-4}$

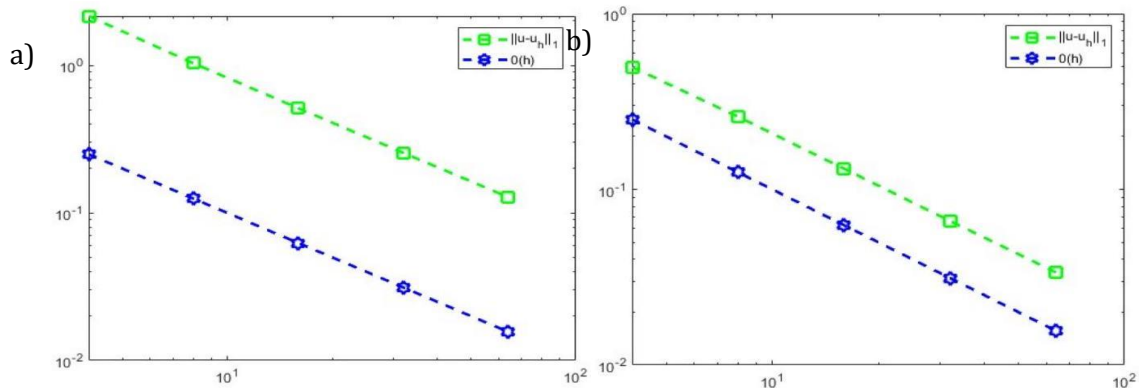


Fig. 3 a) Error and order error of H^1 norm in WGFEM and b) Error and order error of H^1 norm in CADWGFEM.

Conclusions

The performance of the method for solving convection propagation problems is significantly improved by combining classical Artificial Diffusion (CAD) with Weak Galerkin Finite Element (WGFE), known as CADWGFEM. It surpasses classical WGFE in terms of stability, error reduction, and regularity, and is particularly efficient when the diffusion factor is small. This improved strategy is effective in eliminating oscillations and increasing convergence rates under low epsilon conditions. CADWGFEM improves the quality of the solution by limiting

vibrations, but at the same time, it causes considerable additional diffusion. Even though this extra diffusion can increase the total diffusion, it does not significantly reduce the utility of the method in producing more stable and accurate results. CADWGFE is characterized by its ability to balance stability and precision in the solutions it manages.

References

1. Johnson, 1987. Numerical Solution of Partial Differential Equations by the Finite Element Method, Cambridge University Press, Cambridge, England.
2. E. Burman. (2010). Consistent SUPG-method for transient transport problems; stability and convergence, *Comput, Methods Appl, Mech. Eng*, 199 -1114-1123.
3. V. John, J. Nowo. (2011). Error analysis of the SUPG finite element discretization of evolutionary convection-diffusion-reaction equations, *SIAM J. Numer. Anal.* 49 1149-1176.
4. V. John, P. Knobloch. 2007. On spurious oscillations at layers diminishing (SOLD) methods for convection-diffusion equations: Part I - A review, *Computer Methods in Applied Mechanics and Engineering* 196 (17-20): 2197-2215.
5. Kashkool, H.A.2002. Upwind finite element scheme for nonlinear convection-diffusion and application to numerical simulation reservoir, PH. D. thesis.
6. N. Ahmed, G. Mathies. (2016). Numerical study of SUPG and LPS methods combined with high order variational time discretization schemes applied to time-dependent linear convection-diffusion-reaction equations, *J. Sci. Comput.* 67 988-1018.
7. HG. Roos, M. Stynes, L. Tobiska.2008. Robust numerical methods for singularly perturbed differential equations, in *Convection-Diffusion-Reaction and Flow Problems*, second edition, in Springer Series in Computational Mathematics, vol. 24, Springer-Verlag, Berlin.
8. Ab-Adam, W. M. and Kashkool, H. A. (2020). The Error Analysis for the Discontinuous Galerkin Finite Element Method of the Convection-Diffusion Problem. *Journal of Basrh Researches ((sciences))* 45 (2).; pp. 88-106.
9. Saadoon, J. J. and Kashkool, H. A(2019). The error analysis of linearized discontinuous Galerkin finite element mothed for incompressible miscible displacement in porous media, *Journal of Interdisciplinary Mathematics*, 22(8), pp.1471-1484.
10. Kashkool, H. A. and Saadoon, J. J. (2020). Hp-discontinuous Galerkin Finite Element mothed for Incompressible Miscible Displacement in Porous Media, *Journal of Physics: Conference Series*,1530, pp. 1-18.
11. Yanqing Wang, Haiyan Su Xinlong Feng. 2018. Streamline diffusion finite element method for stationary incompressible natural convection problem. *An International Journal of Computation and Methodology NUMERICAL HEAT TRANSFER, PART B: FUNDAMENTALS, VOL. 74, NO. 2*, 519-537.

12. M. Augustin, A. Caiazzo, A. Fiebach, J. Fuhrmann, V. John, A. Linke, R. Umba. (2011) . An assessment of discretizations for convection-dominated convection-diffusion equations, *Comput. Methods Appl. Mech. Eng.* 200 (47-48) 3395-3409.
13. V. John, E. Schmeyster. (2008). Finite element methods for time-dependent convection-diffusion-reaction equations with small diffusion, *Comput. Methods Appl. Mech. Eng.* 198, 475-494.
14. Wang, J. P., and Ye, X. (2013). A weak Galerkin finite element method for second-order elliptic problems, *J. Comput. Appl. Math.* 241(1), pp.103-115.
15. Li, Q.H. and Wang, J., (2013). Weak Galerkin finite element methods for parabolic equations. *Numerical Methods for Partial Differential Equations*, 29(6), pp.2004-2024.
16. Cheichan, M. S., Kashkool, H. A., and Gao, F. (2019): "A weak Galerkin finite element method for solving nonlinear convection-diffusion problems in two dimensions". *Applied Mathematics and Computation*, 354, 149-163.
17. Cheichan, M. S., Kashkool, H. A., and Gao, F. (2019): "A Weak Galerkin Finite Element Method for Solving Nonlinear Convection-Diffusion Problems in One Dimension". *International Journal of Applied and Computational Mathematics*, 5(2), 1-15.
18. Cheichan, M. S., Kashkool, H. A., and Gao, F. (2020): "A weak Galerkin finite element method for solving linear convection-diffusion problems in two dimensions". *Applied of Comput. Anal. and Application*, 28(1), 425-436.
19. Maryam M. Shnawa and Hashim A. Kashkool, 2023 : "Petrov Weak Galerkin Finite Element
20. Method for Solving Coupled Burgers' Problem", *Journal of Utilitas Mathematica.*, 120, 0315-3681.
21. [20] Gao, F., Cui, J., and Zhao, G., (2017). Weak Galerkin finite element methods for Sobolev equation. *Journal of Computational and Applied Mathematics*, 317, pp.188202
22. [21] Quarteroni, A. and Valli, A. 1994. *Numerical Approximation of Partial Differential Equations*, Springer-Verlag, Berlin and Heidelberg, Germany.
23. [22] Eriksson, K., Estep, D., Hansbo, P. and Johnson, C. 1996. *Computational differential equations*, Cambridge University Press, Lund, Sweden.

تحليل الخطأ لطريقة العناصر المحدودة للانتشار الاضطرابي الكلاسيكية لجاليركين الضعيف لمعادلة تفاعل انتشار الحمل الحراري في الحالة الثابتة ثنائية الأبعاد

ابتihal احمد عبد *1، هاشم عبد الخالق كشكول1

1- قسم الرياضيات، كلية التربية للعلوم الصرفة، جامعة البصرة، العراق

الخلاصة:

في هذه الدراسة قمنا بتعديل الخطأ في طريقة جاليركين الضعيفة عند حل المسائل التي يكون فيها الانتشار هو الحمل الحراري السائد $(h)_{L^\infty(\Lambda)}$ في بعدين. ويتم ذلك عن طريق إضافة مصطلح الانتشار الاضطرابي δw ، حيث $\delta = h - \epsilon$. تم عرض طريقة العناصر المحدودة للوظائف المنفصلة باستخدام عامل التدرج المحدد بشكل ضعيف في هذه الدراسة. حيث تم تقديم مفهوم التدرج المنفصل الضعيف، والذي يلعب دوراً هاماً عند استخدام الطرق العددية لحل المعادلات التفاضلية الجزئية. ان الهدف من هذه الدراسة هو تعزيز دقة وثبات الحلول من خلال دراسة خصائص الإهليلجية والثبات للطريقة والتي تعمل على التأكد الطريقة العددية تحتفظ بخصائص المعادلة الأصلية مع تقليل التذبذبات التي تحدث مع طريقة العناصر المحدودة لجاليركين الضعيفة. تم استخدام نظريات محددة لتقدير الخطأ في المعلمات H^1 . و توضح الأمثلة العملية كيف تعمل هذه الطريقة على تحسين التعامل مع المعادلات الجزئية التي تتميز بالانتشار الذي يهيمن عليه الحمل الحراري، مما يعزز قدرتها على تطوير عمليات المحاكاة العددية في الهندسة والفيزياء.

معلومات البحث:

تاريخ الاستلام:

تاريخ التعديل:

تاريخ القبول:

تاريخ النشر:

الكلمات المفتاحية:

الانتشار الاضطرابي الكلاسيكي،

مشكلة تفاعل انتشار الحمل الحراري،

تحليل الأخطاء، الاستقرار

معلومات المؤلف

الابميل:

ibtehalahmedalsady@gmail.com

الموبايل: +9647728407704