IRAQI Academic Scientific Journals

Samarra Journal of Pure and Applied Science

SJPAS

www.sjpas.com

p ISSN: 2663-7405
e ISSN: 2789-6838

Statistical Properties and Parameters Estimation for a New Truncated Lomax Exponential-G Family

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https://doi.org/10.54153/sjpas.2024.v6i4.953

Article Information

Received: 02/07/2024
Revised: 03/08/2024
Accepted: 04/08/2024
Published: 30/12/2024

Keywords:

Lomax exponential , TLE-G family, moments, real-world dataset, order statistic

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Abstract

A novel family of statistical distributions based on the truncated Lomax exponential distribution over the interval [0, 1], called ([0, 1] TLE-G) is proposed in this study. Certain characteristics of the family are derived, among which are the moments, moment-generating function, quantile function, incomplete moments, order statistics, Renyi functions, and Lorenz Bonferroni curves. The truncated Lomax exponential Rayleigh distribution, as a sub-model from the ([0, 1] TLE-G) family, is examined. The flexibility of the new distribution is compared with some other well-known models in the literature by using a real-life dataset. The results of the suggested distribution outperform the other competing distributions.

Introduction

In the world of statistics, statistical families are among the most important tools in statistical analysis. It has a main role in generating statistical distributions, which is useful for analyzing data on various natural and social phenomena. Many methods have been proposed in the literature for generating families of probability distributions. It is noteworthy to mention the Beta method from which the following families of distributions were generated: the Beta-G family (Eugene et al., 2002), Kumaraswamy-G family [1], truncated Weibull-G (TW-G) [2], [0,1] Truncated Lomax-G distribution (Hassan et al., 2020), [0,1] truncated Fréchet-G generator of distributions [3], Truncated Inverted Kumaraswamy (Rashad et al., 2019), [0,1] Truncated Inverse Weibull-G family [4], A New Family of Truncated Nadarajah-Haghighi-G [5,6], the Marshall-Olkin Topp Leone-G family of distributions. [7].

Statisticians devised another method for generating families of distributions by using the Transformed-Transformer method. It is known as the (T-X) method [8]. It was used to find many families, like the Weibull-G family [9], the generalized additive Weibull-G family of distributions [10], the Power Lindley-G family, extended odd Fréchet-G [11], exponentiated T-

X by [12], exponentiated generalized T-X by [13], odd generalized exponential family odd Topp- Leone odd log-logistic-G family and Extended odd Fréchet-G family of distributions.

Many compound distributions have been proposed using these families of distributions. The beta Burr type X distribution [14], Gompertz Flexible Weibull Distribution Gompertz Inverse Exponential distribution [15], Zech distribution, Exponentiated Gompertz Exponential distribution [16], Exponentiated Gompertz Exponential distribution.

In this context, the development of "A New Truncated Lomax Exponential G-Family" (TLE-G), is a qualitative contribution to the field of generating and generalizing statistical distributions. This family was generated based on the exponential Lomax distribution over the period [0, 1]. The developed family has unique characteristics, which allow researchers to find distributions under its tent that can be applied to a wide range of data on different natural phenomena.

The main motive behind this work is to develop a family based on a truncated distribution over the period [0, 1], which represents an important breakthrough in the field of statistical distributions by enhancing the understanding of the generation of distributions with the aim of effectively employing them on a variety of data sets in various fields of life.

The research focused on the use of the (T-X) mechanism to generate a new family of continuous distributions, with the distinctive characteristic of being able to model skewed data. The Lomax exponential distribution proposed by [17-22], has its cumulative distribution function (cdf) and probability density function (pdf) given in equations (1) and (2) respectively.

$$F(x) = 1 - \left[1 + \frac{xexp(x)}{b}\right]^{-a}, \qquad x > 0\,, a, b\, > 0 \qquad (1)$$

$$f(x) = ab^{-1}(1+x)\exp(x)\left[1 + \frac{xexp(x)}{b}\right]^{-(a+1)}, \quad x > 0\,, a, b\, > 0 \qquad (2)$$

In this research, the features and characteristics of the new (TLE-G) family will be explored. The research is carried out under the following methodologies: In the first section, an introduction which gives a brief account of literature review is discussed. In the second part, the proposed family is derived by using the (T-X) method. In the third section, some statistics and mathematics properties of the proposed family are explained. In the fourth section, the method of maximum likelihood is used to estimate the parameters of the proposed family. The fifth section includes the derivation of Truncated Lomax Exponential Rayleigh distribution and its' application to real-world data sets. The concluding note is given in section six.

2. Generating a New Family

A new [0, 1] TLE-G family is generated based truncated Lomax exponential distribution by using T-X method where a random variable $T\sim$ Truncated Lomax distribution with cdf and pdf expressed in equations (3) and (4).

$$R(t) = Þ\left(1 - \left[1 + \frac{texp(t)}{b}\right]^{-a}\right) \quad 0 < t < 1,\ a, b\, > 0 \qquad (3)$$

$$r(t) = Þab^{-1}(1+t)\,exp(t)\,, \quad 0 < t < 1,\ a, b\, > 0 \qquad (4)$$

Where $Þ = \frac{1}{1-\left[1+\frac{exp(1)}{b}\right]^{-a}}$

Now, the method of T-X is employed to generate the TLE-G family as follows:

$F(x) = R(G(x, \boldsymbol{\xi})$

$$F(x) = Þ\left(1 - \left[1 + \frac{G(x,\xi)exp(G(x,\xi))}{b}\right]^{-a}\right) \quad (5)$$

Differentiating equation (5) with respect to x,

$$f(x) = Þab^{-1}g(x, \boldsymbol{\xi})\big(1 + G(x, \boldsymbol{\xi})\big)\, exp\big(G(x, \boldsymbol{\xi})\big)\left[1 + \frac{G(x,\xi)exp(G(x,\xi))}{b}\right]^{-a-1} \quad (6)$$

Where, $0 < x < \infty$, $a, b > 0$. The eq. (5) and (6) represent the cdf and pdf for the TLE-G family where $G(x, \boldsymbol{\xi}), g(x, \boldsymbol{\xi})$ are the cdf and pdf for the baseline distribution and $\boldsymbol{\xi}$ is the parameters' vector. In addition, the survival, hazard, reversed hazard and cumulative hazard functions for the (TLE-G) family are provided in equations (7), (8), (9) and (10) respectively:

$$S(x) = 1 - Þ\left(1 - \left[1 + \frac{G(x,\xi)exp(G(x,\xi))}{b}\right]^{-a}\right) \quad (7)$$

$$h(x) = \frac{Þab^{-1}g(x,\xi)\big(1+G(x,\xi)\big)\, exp\big(G(x,\xi)\big)\left[1+\frac{G(x,\xi)exp(G(x,\xi))}{b}\right]^{-a-1}}{1- Þ\left(1-\left[1+\frac{G(x,\xi x)exp(G(x,\xi))}{b}\right]^{-a}\right)} \quad (8)$$

$$k(x) = \frac{Þab^{-1}g(x,\xi)\big(1+G(x,\xi)\big)\, exp\big(G(x,\xi)\big)\left[1+\frac{G(x,\xi)exp(G(x,\xi))}{b}\right]^{-a-1}}{Þ\left(1-\left[1+\frac{G(x,\xi)exp(G(x,\xi))}{b}\right]^{-a}\right)} \quad (9)$$

$$H(x) = -\ln\left(\frac{1-Þ\left(1-\left[1+\frac{G(x,\xi)exp(G(x,\xi))}{b}\right]^{-a}\right)}{Þ\left(1-\left[1+\frac{G(x,\xi)exp(G(x,\xi))}{b}\right]^{-a}\right)}\right) \quad (10)$$

3. Real properties

Important mathematical and statistical properties of TLE-G family such as expansion of PDF, quantile function, moments, moments generating function, incomplete moments, order statistic, moments of order statistic, inequality measure, mean residual life and entropy are derived.

The expansion of pdf is very important to simplify the computed statistical properties of the (TLE-G) family, such as moments, moment generating functions, incomplete moments, inequality measures, mean and median deviations, mean residual life, stress reliability, and order statistics. This section provides an expansion of the probability density function.

In this subsection, expansions of the pdf of TLE-G family are investigated. Using the generalized binomial theorem and the power series for the exponential function, we obtain

$$\left[1 + \frac{G(x,\xi)exp(G(x,\xi))}{b}\right]^{-a-1} = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\frac{b^{-n}n^{l}}{l!}\binom{-a-1}{n}(G(x,\xi))^{n+l}$$

Then the pdf can be rewritten as

$$f(x) = Þab^{-1}\sum_{s=0}^{1}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\frac{b^{-n}n^{l}}{l!\,k!}\binom{-a-1}{n}\binom{1}{s}g(x,\xi)(G(x,\xi))^{n+l+k+s}$$

$$f(x) = \Psi_{s.k.n.l}\, g(x,\xi)(G(x,\xi))^{n+l+k+s} \quad (11)$$

Where

$$\Psi_{s.k.n.l} = Þab^{-1}\sum_{s=0}^{1}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\frac{b^{-n}n^{l}}{l!\,k!}\binom{-a-1}{n}\binom{1}{s}$$

Theorem 1

The quantile function for TLE-G family is given by

$$W\left(b\left(\left\{u\left(1-\left[1+\frac{exp(1)}{b}\right]^{-a}\right)-1\right\}^{\frac{-1}{a}}-1\right)\right)$$

Proof:

Since the quantile function is the inverse of the cdf, then

$u = F^{-1}(x)$

$F(x) = Þ\left(1-\left[1+\frac{G(x,\xi)exp(G(x,\xi))}{b}\right]^{-a}\right)$

$u = \frac{1}{1-\left[1+\frac{exp(1)}{b}\right]^{-a}}\left(1-\left[1+\frac{G(x,\xi)exp(G(x,\xi))}{b}\right]^{-a}\right)$

$u\left(1-\left[1+\frac{exp(1)}{b}\right]^{-a}\right)-1 = \left[1+\frac{G(x,\xi)exp(G(x,\xi))}{b}\right]^{-a}$

$\left\{u\left(1-\left[1+\frac{exp(1)}{b}\right]^{-a}\right)-1\right\}^{\frac{-1}{a}}-1 = \frac{G(x,\xi)exp(G(x,\xi))}{b}$

$b\left(\left\{u\left(1-\left[1+\frac{exp(1)}{b}\right]^{-a}\right)-1\right\}^{\frac{-1}{a}}-1\right) = G(x,\xi)exp(G(x,\xi))$

$$W\left(b\left(\left\{u\left(1-\left[1+\frac{exp(1)}{b}\right]^{-a}\right)-1\right\}^{\frac{-1}{a}}-1\right)\right) = Q(u) \qquad (12)$$

The quantile function is very useful for finding important features and characteristics of TLE-G family such as central tendency, dispersion, Skewness and kurtosis. Additionally, the median of TLE-G family random variable can be derived from (12) by setting u=0.5

Theorem 2

The rth moment of TLE-G family of distribution is defined as

$$\mu_r = \Psi_{s.k.n.l}\int_0^\infty x^r g(x,\xi)\left(G(x,\xi)\right)^{n+l+k+s}dx$$

Proof:

The r^{th} moment is defined as $\mu_r = \int_0^\infty x^r f(x)dx$

Then, the r^{th} moment of the TLE-G family of distributions can be written as

$$\mu_r = \Psi_{s.k.n.l}\int_0^\infty x^r g(x,\xi)(G(x,\xi))^{n+l+k+s}dx \qquad (13)$$

Theorem 3

The moment generating function for the (TLE-G) family of distributions is given by

$M_x(t) = \frac{t^r}{r!}\Psi_{s.k.n.l}\int_0^\infty x^r g(x,\xi)(G(x,\xi))^{n+l+k+s}dx$

Proof:

Since the moment generating function of a random variable X is defined as follow

$$M_x(t) = E(e^{tx}) = \int_0^\infty \frac{t^r}{r!}x^r f(x)dx$$

Then the r^{th} moment generating function of the TLE-G family of distributions can be written as follow:

$$M_x(t) = \frac{t^r}{r!}\Psi_{s.k.n.l}\int_0^\infty x^r g(x,\xi)(G(x,\xi))^{n+l+k+s}dx \qquad (14)$$

Theorem 4

The characteristic function of the (TLE-G) family of distributions given by the following form

$Q_x(t) = \frac{(it)^r}{r!}\Psi_{s.k.n.l}\int_0^\infty x^r g(x,\xi)(G(x,\xi))^{n+l+k+s}dx$

Proof:

Since the characteristic function of a random variable X is defined as follow:

$$Q_x(t) = E\left(e^{itx}\right) = \int_0^{\infty} \frac{(ti)^r}{r!} x^r f(x)dx$$

Then $Q_x(t)$ of the TLE-G family of distributions can be written as follow:

$$Q_x(t) = \frac{(it)^r}{r!} \Psi_{s.k.n.l} \int_0^{\infty} x^r g(x,\xi)(G(x,\xi))^{n+l+k+s} dx \quad (15)$$

Theorem 5

The incomplete moments of the TLE-G family of distributions given by the following form

$$M_r(y) = \Psi_{s.k.n.l} \int_0^{y} x^r g(x,\xi)(G(x,\xi))^{n+l+k+s} dx$$

Proof:

In distributional analyses, the inequality measures are derived based on the foundational concept of incomplete moment, so the incomplete moments of a random variable X that follows the TLE-G family of distributions is defined as

$$M_r(y) = \int_0^{y} x^r f(x)dx$$

Then the rth incomplete moment of the TLE-G family of distributions can be written as follow:

$$M_r(y) = \Psi_{s.k.n.l} \int_0^{y} x^r g(x,\xi)(G(x,\xi))^{n+l+k+s} dx \quad (16)$$

Theorem 6

The probability-weighted moments for (TLE-G) family of distributions is given by

$$\rho_{m,\eta} = \Upsilon_{i,d,j} \Psi_{s.k.n.l} \int_0^{\infty} x^m \, g(x,\xi)(G(x,\xi))^{n+l+k+sd+j} dx$$

Proof:

The PWMs of a random variable X is defined by

$$\rho_{m,\eta} = E\left(X^m\left(F^{\eta}(X)\right) = \int_0^{\infty} x^m\left(F^{\eta}(x)\right)f(x)dx \quad (17)$$

Then from equation (5), $F^{\eta}(x)$ can be rewritten as

$$F^{\eta}(x) = Þ^{\eta}\left(1 - \left[1 + \frac{G(x,\xi)exp(G(x,\xi))}{b}\right]^{-a}\right)^{\eta}$$

Now, by using the generalized binomial theorem,

$$\left(1 - \left[1 + \frac{G(x,\xi)exp(G(x,\xi))}{b}\right]^{-a}\right)^{\eta} = \sum_{i=0}^{\infty} \binom{\eta}{i} (-1)^i \left[1 + \frac{G(x,\xi)exp(G(x,\xi))}{b}\right]^{-ai}$$

and

$$\left[1 + \frac{G(x,\xi)exp(G(x,\xi))}{b}\right]^{-ai} = \sum_{d=0}^{\infty} \binom{-ai}{d} b^{-d} \, (G(x,\xi))^d exp(dG(x,\xi))$$

With the use of power series for the exponential function,

$$exp(dG(x,\xi)) = \sum_{j=0}^{\infty} \frac{d^j}{j!} (G(x,\xi))^j$$

$$\left(1 - \left[1 + \frac{G(x,\xi)exp(G(x,\xi))}{b}\right]^{-a}\right)^{\eta} = \sum_{i=0}^{\infty}\sum_{d=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^i d^j}{j!\, b^d} \binom{-ai}{d} \binom{\eta}{i} \, (G(x,\xi))^{d+j}$$

Then

$$F^{\eta}(x) = Þ^{\eta} \sum_{i=0}^{\infty}\sum_{d=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^i d^j}{j!\, b^d} \binom{-ai}{d} \binom{\eta}{i} \, (G(x,\xi))^{d+j}$$

let

$$\Upsilon_{i,d,j} = Þ^{\eta} \sum_{i=0}^{\infty}\sum_{d=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^i d^j}{j!\, b^d} \binom{-ai}{d}\binom{\eta}{i}$$

That is

$$F^{\eta}(x) = \Upsilon_{i,d,j}\ (G(x,\xi))^{d+j} \quad (18)$$

Now by inserting equations (11) and (18) in (17),

$$\rho_{m,\eta} = \Upsilon_{i,d,j}\Psi_{s.k.n.l} \int_0^{\infty} x^m\ g(x,\xi)(G(x,\xi))^{n+l+k+sd+j} dx \quad (19)$$

Theorem 7

The probability density function of p^{th} order statistic for TLE-G family of distributions is given by

$$f_{X_{p:n}}(x) = \frac{n!}{(p-1)!\,(n-p)!}\sum_{i=0}^{n-p}(-1)^i\binom{n-p}{i}\left[Þ\left(1 - \left[1+\frac{G(x,\xi)exp(G(x,\xi))}{b}\right]^{-a}\right)\right]^{i+p-1} Þab^{-1}g(x,\xi)\big(1 + G(x,\xi)\big)\, exp\big(G(x,\xi)\big)\left[1+\frac{G(x,\xi)exp(G(x,\xi))}{b}\right]^{-a-1}$$

Proof:

Let $x_1, x_2, x_3, \dots, x_n$ follow the TLE-G family with the cdf and pdf defined in equations (5) and (6) respectively and let $x_{1:n}, x_{2:n}, x_{3:n}, \dots, x_{n:n}$ be the order statistics obtained from this sample. Then the probability density function of p^{th} order statistic $x_{p:n}$ from TLE-G family is obtained from

$$f_{X_{p:n}}(x) = \frac{n!}{(p-1)!\,(n-p)!} f(x)F(x)^{p-1}[1-F(x)]^{n-p} \quad (20)$$

Substituting the cdf and pdf in equations (5) and (6) respectively into (20)

$$f_{X_{p:n}}(x) = \frac{n!}{(p-1)!\,(n-p)!}\sum_{i=0}^{n-p}(-1)^i\binom{n-p}{i}\left[Þ\left(1 - \left[1+\frac{G(x,\xi)exp(G(x,\xi))}{b}\right]^{-a}\right)\right]^{i+p-1} Þab^{-1}g(x,\xi)\big(1 + G(x,\xi)\big)\, exp\big(G(x,\xi)\big)\left[1+\frac{G(x,\xi)exp(G(x,\xi))}{b}\right]^{-a-1}$$

Theorem 8

Lorenz Curve of TLE-G family of distributions can be written as:

$$L_F(x) = \frac{\Psi_{s.k.n.l}}{\mu}\int_0^y x\ \ g(x,\xi)(G(x,\xi))^{n+l+k+s}\, dx$$

Proof:

The Lorenz curve of a random variable can be obtained in the following form:

$$L_F(x) = \frac{1}{\mu}\int_0^y xf(x)dx$$

Substituting the $f(x)$ defined in (11), the Lorenz curve of TLE-G family of distributions is

$$L_F(x) = \frac{\Psi_{s.k.n.l}}{\mu} \int_0^y x \;\; g(x,\xi)(G(x,\xi))^{n+l+k+s}\, dx \qquad (21)$$

Theorem 9

Bonferroni Curve of TLE-G family of distributions can be obtained from:

$$B_F(x) = \frac{\Psi_{s.k.n.l}Þ\left(1-\left[1+\frac{G(x,\xi)exp(G(x,\xi))}{b}\right]^{-a}\right)}{\mu} \int_0^y x \;\; g(x,\xi)(G(x,\xi))^{n+l+k+s}\, dx$$

Proof:

The Bonferroni Curve of a random variable X is defined as $B_F(x) = \frac{L_F(x)}{F(x)}$,

$$B_F(x) = \frac{\Psi_{s.k.n.l}Þ\left(1-\left[1+\frac{G(y,\xi)exp(G(y,\xi))}{b}\right]^{-a}\right)}{\mu} \int_0^y x \;\; g(x,\xi)(G(x,\xi))^{n+l+k+s}\, dx \qquad (22)$$

Theorem 10

The Mean Residual Life (MRL) of TLE-G family of distributions is given by

$$\bar{M}(y) = \frac{\Psi_{s.k.n.l}}{1- \;Þ\left(1-\left[1+\frac{G(y,\xi)exp(G(y,\xi))}{b}\right]^{-a}\right)}\left(\int_{-\infty}^{y} x \;\; g(x,\xi)(G(x,\xi))^{n+l+k+s}\, dx\right) - y$$

Proof:

Since the MRL for a random variable is given by the following general form

$$\bar{M}(y) = \frac{1}{S(y)}\left(\int_y^{\infty} xf(x)dx\right) - y$$

Substituting $f(x)$ and $S(y)$ defined in (11) and (7) respectively,

$$\bar{M}(y) = \frac{\Psi_{s.k.n.l}}{1- \;Þ\left(1-\left[1+\frac{G(y,\xi)exp(G(y,\xi))}{b}\right]^{-a}\right)}\left(\int_{-\infty}^{y} x \;\; g(x,\xi)(G(x,\xi))^{n+l+k+s}\, dx\right) - y \quad (23)$$

Theorem 11

Rényi entropy for the TLE-G family of distributions is given as

$$I_R(q) = \frac{1}{1-q}\log\left\{\int_0^{\infty} (\Psi_{s.k.n.l}g(x,\xi)(G(x,\xi))^{n+l+k+s})^q dx\right\}, q > 0, q \neq 1$$

Proof:

Rényi entropy for random variable X is defined by

$$I_R(q) = \frac{1}{1-q}\log\left\{\int_{-\infty}^{\infty} f^q(x)dx\right\} \qquad q > 0\,, q \neq 1$$

Then the Rényi entropy for a random variable X that follows the TLE-G family of distributions is defined as

$$I_R(q) = \frac{1}{1-q}\log\left\{\int_0^\infty (\Psi_{s.k.n.l}\, g(x,\xi)(G(x,\xi))^{n+l+k+s})^q dx\right\}, q > 0, q \neq 1 \quad (24)$$

Theorem 12

Arimoto entropy for the TLE-G family of distributions is defined as

$$A_q = \frac{q}{1-q}\left(\left(\int_0^\infty (\Psi_{s.k.n.l}\, g(x,\xi)(G(x,\xi))^{n+l+k+s})^q dx\right)^{\frac{1}{q}} - 1\right)$$

Proof:

Arimoto entropy for a random variable X is given as

$$A_q = \frac{q}{1-q}\left(\left(\int_{-\infty}^{\infty} f^q(x)dx\right)^{\frac{1}{q}} - 1\right) , \quad q > 0 , q \neq 1$$

$$= \frac{q}{1-q}\left(\left(\int_0^\infty (\Psi_{s.k.n.l}\, g(x,\xi)(G(x,\xi))^{n+l+k+s})^q dx\right)^{\frac{1}{q}} - 1\right), \quad q > 0 , q \neq 1 \quad (25)$$

Theorem 13

Havrda and Charvat Entropy (HCEN) for the TLE-G family of distributions is given by

$$\frac{1}{2^{1-q}-1}\left(\left(\int_0^\infty (\Psi_{s.k.n.l}\, g(x,\xi)(G(x,\xi))^{n+l+k+s})^q dx\right)^{\frac{1}{q}} - 1\right)$$

Proof:

The Havrda and Charvat Entropy (HCEN) measure is computed from the equation below:

$$HC_q = \frac{1}{2^{1-q}-1}\left(\left[\int_0^\infty f^q(x)dx\right]^{\frac{1}{q}} - 1\right), \quad q > 0 , q \neq 1$$

$$= \frac{1}{2^{1-q}-1}\left(\left(\int_0^\infty (\Psi_{s.k.n.l}\, g(x,\xi)(G(x,\xi))^{n+l+k+s})^q dx\right)^{\frac{1}{q}} - 1\right), q > 0 , q \neq 1 \quad (26)$$

Theorem 14

Tsallis Entropy (TEN) for the TLE-G family of distributions is given by

$$\frac{1}{1-q}\left(1 - \int_0^\infty (\Psi_{s.k.n.l}\, g(x,\xi)(G(x,\xi))^{n+l+k+s})^q dx\right)$$

Proof:

The Tsallis Entropy (TEN) measure for TLE-G family is computed from the equation below:

$$T_q = \frac{1}{1-q}\left(1 - \int_0^{\infty} f^{\eta}(x)dx\right) \quad , \quad q > 0\,, q \neq 1$$
$$= \frac{1}{1-q}\left(1 - \int_0^{\infty} (\Psi_{s.k.n.l}\, g(x,\xi)(G(x,\xi))^{n+l+k+s})^q dx\right), \quad q > 0\,, q \neq 1 \quad (27)$$

4. Maximum Likelihood Method

Suppose that $x_1, x_2, \dots, x_n$ is a random sample of size n from the TLE-G family of distributions. Then the corresponding likelihood function is given by:

$L(a,b,\xi) = \prod_{i=1}^{n} f(x_i, a, b, \xi)$

The log-likelihood function is expressed as

$$L(a,b,\xi) = \prod_{i=1}^{n} \frac{ab^{-1}}{1 - \left[1 + \frac{exp(1)}{b}\right]^{-a}}\, g(x_i,\xi)\big(1 + G(x_i,\xi)\big)\, exp\big(G(x_i,\xi)\big)\left[1 + \frac{G(x_i,\xi)exp(G(x_i,\xi))}{b}\right]^{-a-1}$$

$$L(a,b,\xi) = nlog(a) - nlog(b) - nlog\left(1 - \left[1 + \frac{exp(1)}{b}\right]^{-a}\right) + \sum_{i=0}^{n} g(x_i,\xi) + \sum_{i=0}^{n} log\big(1 + G(x_i,\xi)\big) + \sum_{i=0}^{n}\big(G(x_i,\xi)\big) + (-a-1)\sum_{i=0}^{n} \log\left(1 + \frac{G(x_i,\xi)exp\big(G(x_i,\xi)\big)}{b}\right) \quad (28)$$

5. Truncated Lomax Exponential Rayleigh (TLER) distribution

The TLE-G family can be used to extend different distributions. In this context, Rayleigh distribution serves as the baseline distribution. To find the cdf of TLER distribution, insert the cdf of Rayleigh distribution into equation (5). The pdf of TLER distribution can be found by inserting both pdf and cdf of Rayleigh distribution into equation (6). The cumulative distribution function, probability density function, survival and hazard functions of TLER distribution are provided in equations (31), (32), (33) and (32) respectively. The cdf and pdf of Rayleigh distribution are given in equations (29) and (30) below

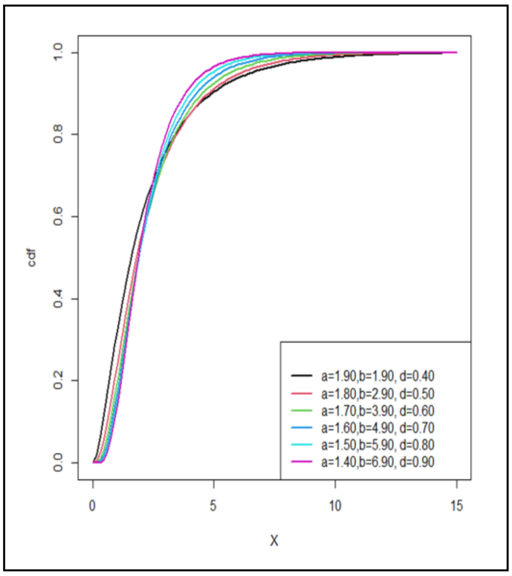
$$\mathrm{F(x)} = 1 - e^{-dx^2} \quad (29)$$
$$f(x) = e^{-dx^2} \quad (30)$$

$$F_{TLER}(x) = Þ\left(1 - \left[1 + \frac{(1-e^{-dx^2})exp\big(1-e^{-dx^2}\big)}{b}\right]^{-a}\right) \quad (31)$$

$$f_{TLER}(x) = Þab^{-1}e^{-dx^2}\Big(1 + \big(1 - e^{-dx^2}\big)\Big)\, exp\big(1 - e^{-dx^2}\big)\left[1 + \frac{(1-e^{-dx^2})exp\big(1-e^{-dx^2}\big)}{b}\right]^{-a-1} \quad (32)$$

$$S_{TLER}(x) = 1 - Þ\left(1 - \left[1 + \frac{(1-e^{-dx^2})exp\big(1-e^{-dx^2}\big)}{b}\right]^{-a}\right)$$

$$h_{TLER}(x) = \frac{\text{Þ}ab^{-1}g(x,\xi)\left(1+\left(1-e^{-dx^2}\right)\right)exp\left(1-e^{-dx^2}\right)\left[1+\left(1-e^{-dx^2}\right)\right]^{-a-1}}{1-\text{Þ}\left(1-\left[1+\frac{\left(1-e^{-dx^2}\right)exp\left(1-e^{-dx^2}\right)}{b}\right]^{-a}\right)} \tag{33}$$



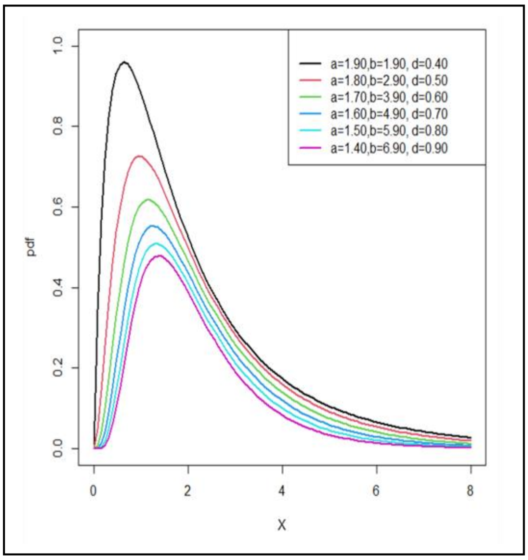
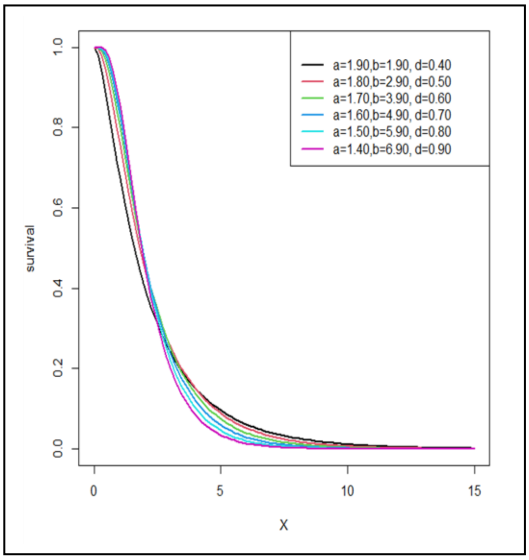


Fig. 1. Cdf of TLER distribution

Fig. 2. Pdf of TLER distribution

From Fig. 1. above, cdf plot of TLER distribution shows that it's a true pdf having ended at 1 for all parameter values and from Fig. 2. The pdf has right skewed with short tail.



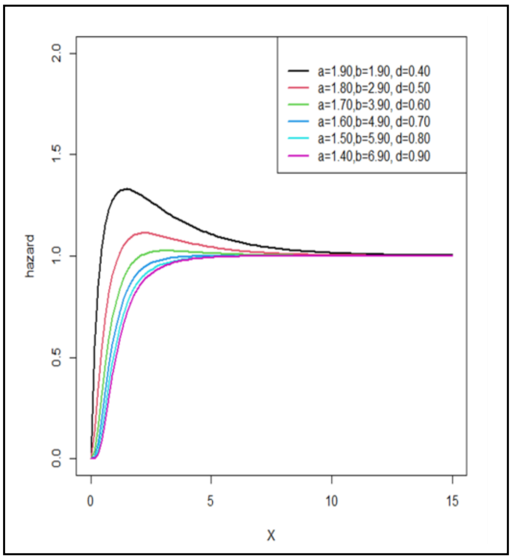


Fig. 3. Survival rate of TLER

Fig. 4. Hazard rate of TLER distribution

According to Fig. 3. the survival rate function of TLER distribution at different parameters is decreasing, while from Fig. 4. the hazard rate function of TLER distribution at different parameters could be increasing or constant.

6. Application

In this section, we fit the TLER distribution to a real data set to showcase its modelling ability. The result is then compared to those of Beta Rayleigh (BeR), Kumaraswamy Rayleigh (KuR), Exponentiated Generalized Rayleigh (EGR), Gompertz Rayleigh (GoR) and Rayleigh distributions. R Statistical Software is used for the analysis. The statistical criteria -*l*, AIC, BIC, HQIC are used to select the best distribution.

The data set given below represents the life of fatigue fracture of Kevlar 373/epoxy that are subjected to constant pressure at the 90% stress level until all had failed. The data has been used by [23-27].

0.0251 0.0886 0.0891 0.2501 0.3113 0.3451 0.4763 0.5650 0.5671 0.6566 0.6748
0.6751 0.6753 0.7696 0.8375 0.8391 0.8425 0.8645 0.8851 0.9113 0.9120 0.9836
1.0483 1.0596 1.0773 1.1733 1.2570 1.2766 1.2985 1.3211 1.3503 1.3551 1.4595
1.4880 1.5728 1.5733 1.7083 1.7263 1.7460 1.7630 1.7746 1.8275 1.8375 1.8503
1.8808 1.8878 1.8881 1.9316 1.9558 2.0048 2.0408 2.0903 2.1093 2.1330 2.2100
2.2460 2.2878 2.3203 2.3470 2.3513 2.4951 2.5260 2.9911 3.0256 3.2678 3.4045
3.4846 3.7433 3.7455 3.9143 4.8073 5.4005 5.4435 5.5295 6.5541 9.0960.

	n	mean	median	min	max	skew	kurtosis
Data	7**6**	1.96	1.**74**	0.03	9.1	1.94	4.95

Table 1: A descriptive statistics for Data.

From Table 1, it is glaring that the data is positively skewed, and its peakedness is above that of normal distribution.

Distribution	W	K-S	p-value
TLER	0.1028	0.0755	0.7494
BeR	0.2054	0.1465	0.0687
KuR	0.1495	0.1190	0.2139
EGR	0.2220	0.1551	0.0459
GoR	0.6640	0.0890	0.5523
R	0.2086	0.2043	0.0029

Table 2: The K-S value with its corresponding p-value and W value of Data

From Table 2, it can be inferred that TLER distribution, having the least values of Komolgorov-Smirnov (K-S) and Shapiro-Wilk (W) statistics together with the highest p-value, is the best among the competing models.

Distribution	-2L	AIC	CAIC	BIC	HQIC
TLER	122	251	251	258	253

BeR	124	255	256	262	258
KuR	123	252	252	259	255
EGR	125	257	257	263	259
GoR	125	257	258	264	260
R	137	276	276	278	277

Table 3: The values of -2L, AIC, CAIC, BIC, HQIC for the Data.

From table 3, TLER has the least values of –LL, AIC, CAIC, BIC, and HQIC and is therefore considered as the best of all the competing models.

Distribution	$\hat{a}$	$\hat{b}$	$\hat{d}$
TLER	1.1742	0.0964	0.02963
BeR	0.5401	3.52940	0.0258
KuR	0.6351	8.5800	0.0066
EGR	0.7408	0.5324	0.1352
GoR	7.6551	1.5268	0.0332
R	0.1591	–	–

Table 4: Estimation parameters using MLE with respect to the given dataset.

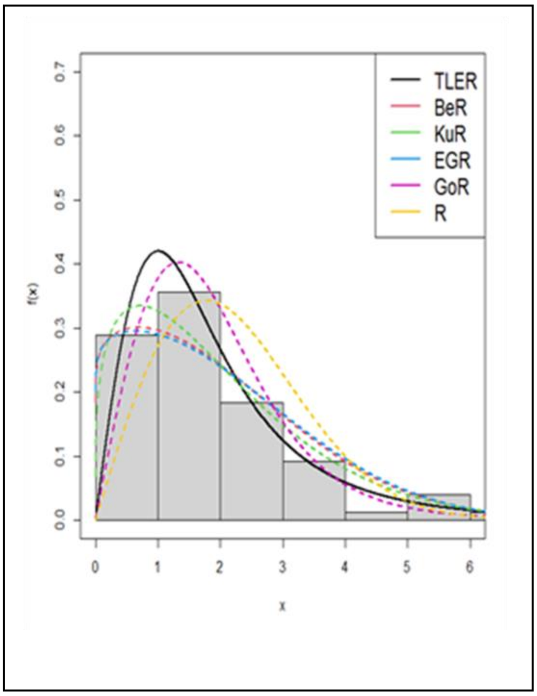


Fig. 5. Estimated pdf of distribution for Data.

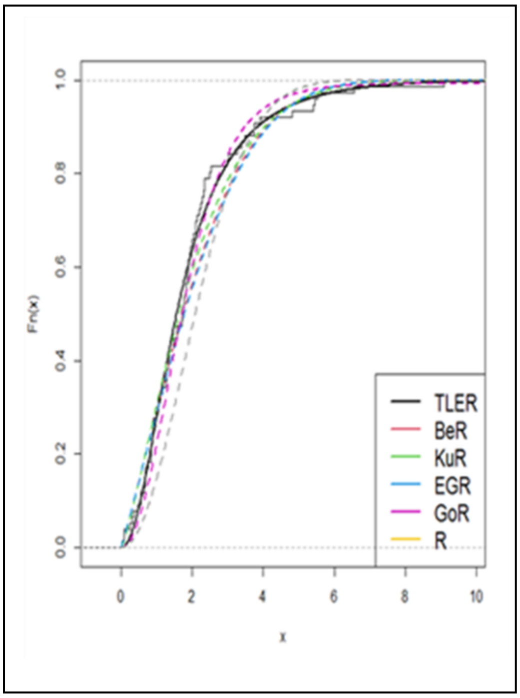


Fig. 6. Estimated cdf of distribution for Data

From Fig. 5. it can be observed that TLER distribution fits the data best. From Fig. 6.it can be inferred that TLER distribution's fitted cdf fits the empirical cdf (the black stepped line) better than the other distributions.

7. Conclusions

Flexibility in generalized models plays an important role and is a strong motivation for developing new sets of distributions. This study presented a new family, called TLE-G family that was built on a new truncated distribution. The Statistical properties such as the quantile function, order statistics, moments, Lorenz and Bonferroni curves, and four different kinds of entropies were proposed for the newly derived family. The maximum likelihood method is used to estimate the parameters of the new family. A special distribution of the TLE-G family was developed to demonstrate the flexibility of the family by using real dataset.

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www.sjpas.com

p ISSN: 2663-7405
e ISSN: 2789-6838

الخصائص الإحصائية و تقدير المعلمات لعائلة لوماكس الاسي المبتور

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معلومات البحث:

تأريخ الاستلام: 2024/07/02

تاريخ التعديل : 2024/08/03

تأريخ القبـــول: 2024/08/04

تاريخ الـنشر: 2024/12/30

الكلمات المفتاحية:

لوماكس الاسي ، عائلة لوماكس الاسي المبتور ، العزوم 3، مجموعة بيانات حقيقية ، الاحصاءات المرتبة

معلومات المؤلف

الايميل:

الموبايل:

الخلاصة:

تقدم هذه الدراسة عائلة جديدة من التوزيعات الإحصائية تستند إلى توزيع لوماكس المبتور على فترة [0، 1] وتسمى ([0، 1] TLE-G). تم استنتاج بعض الخصائص لهذه العائلة، بما في ذلك االعزوم، ودالة توليد العزوم، و الدالة الكمومية، و العزوم الناقصة، و الإحصاءات المرتبة، ودوال ريني، ومنحنيات لورنتز بونفيرني. تم فحص توزيع لوماكس الاسي المبتور رايلي ، كنموذج فرعي من عائلة ([0، 1] TLE-G). تمت مقارنة مرونة التوزيع الجديد مع بعض النماذج المعروفة الأخرى في الأدبيات باستخدام مجموعة بيانات حقيقية. أظهرت نتائج التوزيع المقترح تفوقًا على التوزيعات المنافسة الأخرى